Games, Economies and Fair Allocations

Marc Meertens
Games, Economies and Fair Allocations

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Contents

Acknowledgements i

1 Introduction 1
   1.1 Game Theory .............................................. 1
   1.1.1 Non-cooperative Game Theory .......................... 2
   1.1.2 Cooperative Game Theory ............................... 4
   1.2 Outline of the thesis ..................................... 6

2 Bargaining Sets in Cooperative Games 11
   2.1 Introduction ............................................. 11
   2.2 Transferable utility games ............................... 14
   2.3 Bargaining sets in TU-games .............................. 19
   2.4 Bargaining sets and the core ............................ 22
   2.5 Symmetric TU-games ...................................... 25
     2.5.1 The reactive and the semireactive bargaining set 25
     2.5.2 The bargaining set .................................. 27
     2.5.3 Summary ............................................. 34
   2.6 Examples ................................................ 34

3 Bargaining Sets and Price Equilibria in Exchange Economies 43
   3.1 Introduction ............................................. 43
   3.2 Preliminaries and some basic results .................... 45
   3.3 Bargaining sets in exchange economies ................... 50
     3.3.1 Existence results .................................. 51
     3.3.2 Bargaining sets and the core ....................... 55
   3.4 Price equilibria in exchange economies ................... 61

4 Envy-Freeness and Pareto Efficiency 67
   4.1 Introduction ............................................. 67
   4.2 Preliminaries ............................................. 71
## Contents

4.3 Envy-free allocations ................................. 73
4.4 Envy-free and Pareto efficient allocations ................................. 76
  4.4.1 Envy-free and Pareto efficient allocations in general .................. 77
  4.4.2 Sufficient conditions for compatibility ...................... 79
4.5 Proof of Theorem 4.3C .................................. 83

5 The Nucleolus of Trees with Revenues 95
  5.1 Introduction ........................................... 95
  5.2 The tree with revenues and the associated game ............... 98
    5.2.1 Definitions and notations .................................. 98
    5.2.2 The nucleolus of a TU-game .............................. 99
    5.2.3 The reduced tree with revenues and the associated game .... 101
    5.2.4 Standard trees ........................................ 103
  5.3 The nucleolus of trees with revenues .......................... 107
    5.3.1 Excesses at the nucleolus ................................ 107
    5.3.2 Computing the nucleolus ................................. 112
    5.3.3 Monotonicity properties ............................... 113
    5.3.4 A characterization .................................... 117
    5.3.5 Example ............................................. 123
  5.4 Proofs ............................................... 125

6 Processing Games ........................................ 131
  6.1 Introduction ........................................... 131
  6.2 Processing problems ..................................... 133
  6.3 Processing games ....................................... 137
  6.4 Proof of Theorem 6.4 ................................... 142
  6.5 Final remarks ......................................... 148

7 Dynamic Selection in Normal-Form Games 151
  7.1 Introduction ........................................... 151
  7.2 Normal-form games ..................................... 154
    7.2.1 Preliminaries .......................................... 154
    7.2.2 The Nash-equilibrium .................................... 155
  7.3 Evolution and the Nash-equilibrium ............................. 157
  7.4 The dynamics .......................................... 159
  7.5 The limit set and stability .............................. 161
    7.5.1 The limit set and stability in zero-sum games .......... 167
1

Introduction

1.1 Game Theory

The monograph lying before you deals with several subjects in the field of Game Theory. Broadly speaking, Game Theory is a mathematical theory to model and analyze so-called conflict situations. In a conflict situation a group of goal-seeking individuals, each endowed with his own knowledge, capacities, behavior, likes and dislikes, interact and thereby jointly generate an outcome. Game Theory is a rather young field of study, its foundation was made in an article by John von Neumann in 1928, but the theory received widespread attention only after the publication of the seminal book by Von Neumann and Morgenstern in 1944. Since then, Game Theory is increasingly being applied in a wide variety of disciplines, like economics, auction theory, political sciences, management, behavioral psychology and other social sciences and in evolution theory. From these examples it is clear that the notion of ‘game’ should not be taken too restrictive, since Game Theory goes further than parlor games like chess or poker. Nevertheless, a model of a conflict situation is usually called a game and the individuals or decision-makers involved are called the players of the game.

The game (model of the conflict) itself consists of a number of data such as the number of players and each player’s characteristics, together with the rules of the game. Numerous other features of the real-life situation may be included in the data of a game. Furthermore, there are some basic assumptions concerning the characteristics of the players. First of all, it is quite often assumed that the data of the game is common knowledge. This means that all players know the data of the game, each of the players knows that all players know the data of the game and so on. Second, it is assumed that players understand the game, are capable to express preferences for possible outcomes and that each player tries to obtain among the outcomes the most profitable outcome. This assumption is known as a mild form of rationality.
Game Theory aims to prescribe what each player in a game can do, in order to promote his interest optimally assuming that everyone else behaves rationally as well. To be more precise, Game Theory tries to find solution rules for certain classes of games. Solution rules provide recommendations to the players involved, telling each of the players how to attain an optimal outcome. Before one can try to come up with a certain concept for a solution rule, it first should be clear which approach the players choose to achieve a certain outcome. Roughly speaking, there are two different approaches to attain a particular outcome, namely, players either ‘cooperate’ or ‘compete’ with each other. Traditionally, this division is also reflected in Game Theory. The competitive nature of interaction is the topic of Non-cooperative Game Theory. Here, the players cannot make binding agreements (as well as other commitments) and are considered as individual (expected) utility-maximizers playing against each other. The main focus in Non-cooperative Game Theory is on formalizing the notions of rational behavior and the concept of equilibrium. We should emphasize that the term ‘non-cooperative’ does not mean that this branch of Game Theory is incapable of explaining cooperation within groups of individuals. Rather it focuses on how cooperation may emerge as rational behavior in the absence of the possibility to make binding agreements. In case the players can attain particular outcomes for themselves through binding agreements, we are dealing with Cooperative Game Theory. In this branch of Game Theory it is not only interesting to know how players cooperate in an optimal way, but also the problem of how to allocate the proceeds of the cooperation plays an important role.

1.1.1 Non-cooperative Game Theory

In a non-cooperative game the strategic aspects are all important. Each player involved recognizes his partial influence on the situation, since the outcome does not only depend on his own actions but also on the actions of other players involved in the game. Those strategies of a player that are, according to his preferences, the most profitable to him may depend on strategies that his opponents have already taken, on those he expects them to be taking at the same time and even on future strategies that they may take, or decide not to take, as a result of his current strategy. The main interest within this branch of Game Theory is on finding a set of recommendations that tell each player, in every situation that may arise, how to behave in such a way that benefits him most. By rationality this means that none of the players should be able to improve by deviating unilaterally from such a recommendation. Hence, a recommendation must be self-enforcing, meaning that for each player it should not be in his interest to deviate as long as his opponents follow their recommendations. In game theoretic terminology this means that such a recommendation is a Nash-equilibrium (Nash (1950)), i.e., a strategy profile consisting of one strategy for each player with the property that it is this player’s best response to the strategies actually played by his opponents.

The basic model of a non-cooperative game is the so-called normal-form game which in case of two players is also referred to as a bi-matrix game. Informally, in a normal-form game there are a finite number of players and each of them has a finite number of strategies at his disposal by which he can influence the outcome of the game. Each player chooses, simultaneously and without any possibility or wish to communicate with the other players, a strategy that generates immediately a certain payoff to each one of the players. This ends the
1.1 Game Theory

game. Let us have a look at an example of a bi-matrix game and of a Nash-equilibrium in this game.

**Example.** Imagine that two very good friends have won a bet with a bookmaker and earned 50 euro with it. However, the bookmaker is sly and wants to earn his money back. To do so, he challenges the two friends to divide the money as follows. Each of the two friends can choose, at the same time and without telling the other, either ‘to share’ or ‘not to share’ the 50 euro. In case both friends choose ‘to share’, both of them receive 50 euro. So, in this case the bookmaker pays each friend 50 euro. In case one of them chooses ‘to share’ and the other one chooses ‘not to share’, the latter receives the 50 euro from the bookmaker and receives another 50 euro from the first player (who chooses ‘to share’). However, in case both of them choose ‘not to share’, they will receive nothing and thus the bookmaker will win back his 50 euro. Since, these two friends are convinced of their friendship and loyalty to one and other, they decide to play the game as proposed by the bookmaker.

This game can be modeled as a bi-matrix game in which both players have two strategies ‘to share’ denoted by $S$ and ‘not to share’ denoted by $NS$. The evaluation of both players for all the possible outcomes can be reflected in the following bi-matrix (which explains the name of the game):

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S$</td>
<td>$NS$</td>
</tr>
<tr>
<td><strong>$S$</strong></td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td><strong>$NS$</strong></td>
<td>100</td>
<td>-50</td>
</tr>
</tbody>
</table>

Player 2

Here, Player 1 is the *row player* and Player 2 is the *column player*. If, for instance, Player 1 chooses $NS$ and Player 2 chooses $S$, then the outcome is the entry determined by these choices, meaning that Player 1 receives 100 euro and Player 2 pays 50 euro.

Suppose one gives the recommendation $(S, S)$ as a solution of the game, meaning that Player 1 as well as Player 2 should choose $S$. Then in this situation it is in the interest of Player 1 to deviate from this recommendation, i.e., choose $NS$ instead of $S$, as long as Player 2 follows his recommendation, i.e., still chooses $S$. The reason for Player 1 to choose $NS$ instead of $S$ is because he will get 100 euro instead of 50 euro. Also Player 2 is inclined to deviate from $(S, S)$. If one gives the recommendation $(NS, S)$ as a solution, it is in interest of Player 2 to deviate, i.e., to choose $NS$, since he will get 0 euro instead of $-50$ euro. Similarly, Player 1 is inclined to deviate from $(S, NS)$. However, if one gives the recommendation $(NS, NS)$ it is in no player’s interest to deviate unilaterally from it and therefore the strategy profile $(NS, NS)$ is a Nash-equilibrium. The following figure illustrates this reasoning.
Hence, according to the Nash-equilibrium both friends do not share the money and thus the bookmaker will win his 50 euro back.

Remark. The bi-matrix game discussed in the example is a version of the so-called prisoner’s dilemma. The title prisoner’s dilemma and the rather entertaining version with prison sentences as payoffs are due to Albert W. Tucker (for an explanation see e.g., page 94–95 in Luce and Raiffa (1957)). This bi-matrix game has become the classic example of a non-cooperative game in economics, political and social sciences, and of course in Game Theory.

1.1.2 Cooperative Game Theory

In a cooperative environment the players may or must agree (e.g., enforced by a legal contract) to a joint action that generates a certain outcome. By doing so, they may attain, according to their preferences, a (relatively) high outcome in the given situation. In this branch of Game Theory the strategic aspects become less important while the negotiations are all important. The central question in Cooperative Game Theory is how one can divide the proceeds of cooperation among the players, in such a way that, according to their preferences, is considered to be as fair as possible while taking into account all the contributions of the players to this outcome.

The problem of fair division is as old as the hills. Already in the Hebrew Bible, the issue of fairness is raised in some of the best-known narratives. Fairness triumphed, when King Solomon proposed to cut a baby, claimed by two mothers, in two. When the true mother protested and offered the baby to the other ‘mother’, the truth about the baby’s maternity became apparent. Solomon’s proposed solution is maybe the first explicit mention of fair division, known in recorded history. Although, it is not a real solution (Solomon had no intention of cutting the baby in two), it distinguished the mother from the impostor. This story, among several other stories from the Hebrew Bible, is studied from a game theoretical point of view in Brams (1980).

The first problem of fair division is which normative concept one should use, since it may not be clear which concept suits a division problem best. As a result, many concepts for solution rules can be found in Cooperative Game Theory. Besides, Cooperative Game Theory comprises several different models. The Transferable Utility (TU) game is studied the most. This model of a cooperative game was already introduced in Von Neumann and Morgenstern...
Since then it forms the main pillar of Cooperative Game Theory. The second main model in Cooperative Game Theory is the Non-Transferable Utility (NTU) game and is introduced by Aumann and Peleg (1960). Let us have a look at a cooperative setting from which these two models may arise.

**Example.** Imagine that the two friends in the example of Subsection 1.1.1 decide, after being aware of the ‘catastrophic’ outcome according to the Nash-equilibrium, to make binding agreements before playing the game proposed by the bookmaker. Then the nature of the game changes completely. Let us consider what each coalition, i.e., a non-empty subset of the set of players, can achieve in this example. Player 1 can guarantee himself at least a payoff of zero by choosing $S$, indeed, if he chooses this strategy, he receives either 100 euro (Player 2 plays $N$) or 0 euro (Player 2 plays $S$), while if he chooses $N$ he might have to pay 50 euro. Also Player 2 can guarantee himself a payoff of zero, by choosing $N$. Both players together, i.e., coalition $\{1, 2\}$, can achieve all outcomes within the set $\{(50, 50), (100, -50), (-50, 100), (0, 0)\}$. This situation is depicted in the following figure.

The outcome generated by the two players, will be determined by negotiations. Player 1 will not agree with an outcome below the horizontal line through the point $(0, 0)$, while Player 2 will not agree with an outcome left of the vertical line through the point $(0, 0)$, since both players can guarantee themselves a payoff of zero. Hence, only outcomes in the gray area of Figure 2 are taken into consideration and thus the options are $(0, 0)$ or $(50, 50)$. Since $(50, 50)$ is for both players better than $(0, 0)$ they will, according to rationality, agree to share the 100 euro. Recall that in the non-cooperative setting both players decide, according to the Nash-equilibrium, not to share and end up with nothing i.e., in the point $(0, 0)$.

The cooperative game described above is called a Non-Transferable Utility (NTU) game. In such games, it is assumed that payoffs cannot be transferred between the players. Therefore, in such a game the only important issue for a coalition of players is its set of attainable outcomes. However, since the possible outcomes are expressed in money, both players could also make agreements in which side payments or monetary compensations are used, i.e., the players can agree to choose a certain outcome under an additional condition in which both players pay each other a certain amount of money. For instance, both players can agree to play $(S, N)$ under the condition that Player 2 pays 60 euro to Player 1, yielding an outcome
of \((-50 + 60, 100 - 60) = (10, 40)\). If in addition we assume that both players have an equal evaluation for money, this second approach leads to a Transferable Utility (TU) game. Following the same reasoning as before, both players can guarantee themselves at least the payoff of zero. However, the players together can agree to achieve many more outcomes than in the previous situation, as the following figure illustrates.

![Figure 3. All possible outcomes in case of side payments.](image)

All outcomes within the gray part of Figure 3 can be realized by negotiations between the two players. Observe that the payoff vector of an outcome is not relevant, but only the sum of its coordinates is all important. Hence, by rationality, the only important issue for each coalition is the value of the most valuable outcome which can be realized. In case of coalition \{1, 2\} this value is 100 euro. So, the outcome that will be realized in case both players come to an agreement, is on the dark line in Figure 3, known as the Pareto boundary. This boundary, introduced in Pareto (1909), is used in economic theory in case of decisions with multiple conflicting objectives to obtain rational solutions (see e.g., Keeney and Raiffa (1976)). It is not clear which outcome on this Pareto boundary both players agree to realize. However, the outcome (50, 50) is still one of the possibilities and for this two-person TU-game it is also the standard solution (see e.g., Aumann and Maschler (1985)).

### 1.2 Outline of the thesis

In this monograph we are dealing with both, Non-cooperative and Cooperative Game Theory. Herewith, we give an overview of the subjects studied in this thesis and we briefly discuss the results we have obtained.

**Chapter 2** deals with Cooperative Game Theory. The first part is introductory. It recalls some results and well-known solution concepts in the theory of Transferable Utility (TU) games, one of them being the core (Gillies (1959)). Although the core seems to be a very natural solution concept, its existence cannot always be guaranteed. The bargaining set, introduced by Aumann and Maschler (1964), is an extension of the core. This set contains allocations which do not allow for objections to be justified. Since, a core allocation does not allow any objection, the bargaining set contains the core. However, since the bargaining set also
1.2 Outline of the thesis

contains the kernel (Davis and Maschler (1965)) its non-emptiness can be guaranteed as long as the TU-game has a non-empty imputation set. More recently, Granot (1994) presented a new type of bargaining set, the reactive bargaining set. Also the reactive bargaining set is a non-empty set-valued solution concept containing the kernel and the core. It is closely related to the bargaining set of Aumann and Maschler, but it is a (proper) subset of the bargaining set. The semireactive bargaining set (Sudhölter and Potters (2001)) takes a middle position between these two concepts. It contains the reactive bargaining set and it is a (proper) subset of the bargaining set.

In the main part of the chapter these three type of bargaining sets are studied on the class of symmetric TU-games. It turns out that for a symmetric TU-game the reactive bargaining set is the union of the kernel and the core. If one additionally assumes superadditivity, the same statement holds for the semireactive bargaining set. For the bargaining set and the core of a symmetric TU-game to coincide, one needs total balancedness. By means of examples, it is shown that these assumptions cannot be omitted. Furthermore, the chapter provides a proof for the core and bargaining set of an arbitrary TU-game to coincide whenever the value for the grand coalition is large enough. In other words, if the proceeds of cooperation of all the players is large enough, each of the three bargaining sets coincides with the core. However, we end this chapter by providing an example of a symmetric TU-game which illustrates a rather surprisingly phenomenon of the bargaining set. In the example the bargaining set and the core coincide for all possible values for the grand coalition, except for exactly one value. There, the bargaining set is the union of the core with a finite number of imputations outside the core. This illustrates that the property ‘the bargaining set and the core coincide’ is not a prosperity property (Gellekom, Potters, Reijnierse (1999)).

Chapter 3 and Chapter 4 study fair (re)allocations in economies with indivisible goods and money.

The contents of Chapter 3 is twofold. The first part is closely related to Chapter 2. Here, the reactive, semireactive and bargaining set are studied for exchange economies with indivisible goods and money. In these Debreu-type of economies (Debreu (1959), see also Debreu (1983)) players, in this context usually referred to as agents, can exchange indivisible goods with each other. Payments are made by a perfectly divisible good (money). Agents have preferences for bundles consisting of a number of indivisible goods and an amount of money. These preferences are assumed to be complete and transitive binary relations on the set of consumption bundles. Since, the definitions of the three bargaining sets, studied in the previous chapter for (symmetric) TU-games, are given in ordinal terms (i.e., without explicit reference to utility functions), they can be extended to these types of economies. This is done in the first part of Chapter 3.

We prove that the reactive bargaining set is non-empty, if the preferences of the agents can be represented by quasi-linear utility functions and if the initial endowments satisfy the Total Abundance (TA) condition. The latter is a condition on the amounts of money agents initially have. By examples it is shown that for guaranteeing non-emptiness of the reactive bargaining set neither the TA-condition nor the quasi-linearity of the utilities can simply be omitted. Furthermore, it turns out that reallocations in the (semi)reactive bargaining set are
individual rational (i.e., agents receive bundles which they appreciate weakly better than their initial endowments). This result does not hold for the bargaining set. An exchange economy with quasi-linear utilities generates a non-negative and superadditive TU-game. We show that the core and the (semi)reactive bargaining set of an exchange economy, satisfying the TA-condition, coincide if and only if the core and the (semi)reactive bargaining set of the corresponding TU-game coincide.

The observation that every non-negative and superadditive TU-game generates an exchange economy is the starting point for the second part of the chapter. Here, we prove that if in addition this TU-game is balanced, then it generates an exchange economy with a price equilibrium. However, different economies may yield the same TU-game. So, it is possible that the set of price equilibria of two exchange economies, yielding the same TU-game, may be different. This illustrates that the concept of price equilibrium is not a game theoretical solution concept, i.e., it is a solution rule for economies, not for the corresponding games.

Chapter 4 is closely related to the previous chapter. Here, we also study economies with indivisible goods. However, in this chapter the agents do not have initial endowments, i.e., there is no a priori allocation of the indivisible goods. The problem that arises is how to allocate the indivisible goods among the agents in a fair way. Since these goods cannot be divided, monetary compensations are used to obtain fair allocations. Like in Chapter 3, agents are assumed to have preferences defined on the set of bundles consisting of indivisible goods and money. In this chapter, however, more general preferences than quasi-linear ones are considered.

The first concept, studied in this chapter, for a fair allocation is envy-freeness (Foley (1967)). In an envy-free allocation each agent receives, according to his preference, the best bundle among all the bundles in the allocation. So, each agent thinks that he receives the best part of the allocation. Given that the preferences satisfy strict monotonicity and continuity in money and the archimedean property (i.e., money can change preferences), we prove the existence of envy-free allocations. An example illustrates that continuity in money and the archimedean property cannot be omitted.

From a given envy-free allocation it is still possible for the agents to improve, by reallocating the indivisible goods and the amounts of money. This observation brings us to another concept for allocating the indivisible goods, namely, envy-freeness and Pareto efficiency. A Pareto efficient allocation is optimal for all agents, in the sense that no agent can improve his position without making at least one other individual worse off. We provide an example which illustrates the incompatibility of envy-freeness and Pareto efficiency. Intuitively, this means one has to choose either an allocation which is envy-free, but agents still can improve or an allocation from which nobody can improve without hurting the others, but which causes envy. The final section of the chapter is devoted to provide some additional conditions for the compatibility of envy-freeness and Pareto efficiency.

Chapter 5 and Chapter 6 deal with fair allocations in certain resource sharing problems. A natural approach to solve such problems is to model them as cooperative games and use a solution concept to allocate the costs and/or profits of the shared resource.
Chapter 5 studies the fair allocation of the total profit within a tree with revenues. Here, a tree network is given with non-negative costs on the edges and a number of players in each node. Each player may earn some non-negative revenue, if he is connected to the root of the network (e.g., a central supplier). A cooperating coalition of players is assumed to connect those members to the root who together yield the highest net profit (i.e., the maximum of their total revenues minus the construction costs). The problem that arises is how to allocate the proceeds of this cooperation among the players in a fair way. To study this problem we construct a TU-game and propose the nucleolus (Schmeidler (1969)) as a fair allocation of the total profit. The nucleolus is a single-valued solution concept which for the TU-games studied in this chapter is contained in the core. Calculating the nucleolus in general requires an exponential number of computations, since the solution must adhere to coalitional rationality conditions. The main focus of Chapter 5 is to bypass all the computational complexity and devise a simple algorithm for computing the nucleolus of a tree with revenues. Furthermore, we present some monotonicity properties satisfied by the nucleolus and a characterization of the nucleolus as a solution rule for trees with revenues.

In Chapter 6 the problem of fairly allocating the minimal joint costs of a processing problem is studied. In a processing problem there is a finite number of jobs that need to be completed. Each job requires a specific amount of effort (e.g., money, labor or demand). The main restriction is the available capacity constraint to process jobs. During the time a job is uncompleted a fixed cost (the cost-coefficient of the job) has to be paid. The objective is to minimize the total costs. To do so, the jobs can be processed in any way one likes, there are no restrictions whatsoever on the processing schedule. However, as it turns out the total costs can be minimized by performing the jobs one by one. With this observation one can find easily an efficient algorithm to solve a processing problem. By applying Smith’s rule (Smith (1956)) the optimal order can be derived.

Consider now the situation in which each job is owned by a different player. Each player has an individual capacity to process jobs. It is an upper bound for the amount of effort per time unit he can generate for handling jobs. A coalition of cooperating players faces a processing problem with the coalitional capacity being the sum of the individual capacities of the members. Next to the problem of minimizing the total costs, there is the problem of allocating these costs among the players in a fair way. To analyze this problem we introduce processing games which are TU-games considering costs instead of rewards. The main result of the chapter is the total balancedness of this class of games. To be more precise, we provide an explicit core allocation of the costs. It has the appealing property that the contribution a player has in the total costs is independent on the optimal order that is chosen to process the jobs.

The final chapter of this thesis is a topic in Non-cooperative Game Theory. The main solution concept for non-cooperative games is the Nash-equilibrium (Nash (1950)). However, this concept has several drawbacks. The first one is its multiplicity and the inconsistency of many Nash-equilibria with the intuitive notion about what an equilibrium should be. Moreover, the question remains on how players come to play a Nash-equilibrium. An approach to study these problems was inspired by the theory of evolution of populations of animals. The paper by Maynard Smith and Price (1973) was the starting point of the idea of introducing evolution
of species for a way to justify the concept of Nash-equilibrium.

In Chapter 7 we formalize this idea for normal-form games, by introducing a dynamic selection process defined by differential equations. These differential equations are defined by so-called regret-functions which measure the ‘regret’ of a player, expressed in his payoff, of not having played a certain pure strategy to his disposal. This dynamic selection process has the appealing property that its rest-points and the set of Nash-equilibria coincide. The main question studied is whether the limit set is a subset of the set of Nash-equilibria. Intuitively, this means that the dynamics has a tendency only towards Nash-equilibria. Due to an example of Hart and Mas-Colell (2003) this question has a negative answer. However, the main results of the chapter state that on the class of zero-sum games (Von Neumann (1928)) (or strategic equivalent games) as well as on the class of potential games (Monderer and Shapley (1996)) every limit set is a subset of the set of Nash-equilibria. We provide for each class of normal-form games a Lyapunov function and we prove that both classes of normal-form games require different functions to be Lyapunov functions. Also the concept of asymptotic stability is studied. In a zero-sum game the (unique) Nash-component is asymptotically stable and in a potential game a smoothly connected component of Nash-equilibria is asymptotically stable if and only if it is a local maximizer of the potential.
2

Bargaining Sets in Cooperative Games

2.1 Introduction

For a greater part of the current chapter the bargaining set, the semireactive bargaining set and the reactive bargaining set are studied within the class of symmetric TU-games. Most of the results derived in this part can also be found in Meertens, Potters and Reijnierse (2006b). The first part of the chapter is an introduction to the theory of Transferable Utility games.

In 1964, Aumann and Maschler introduced the idea to base a solution concept for cooperative games on certain bargaining possibilities of the players. They introduced several types of bargaining sets. The variant $M_1^*$, simply referred to as the bargaining set and studied in Davis and Maschler (1967), attracted the most attention. The bargaining set is a set-valued solution concept. Set-valued solution concepts usually recommend which outcomes should be rejected. They do not claim that every element in the set is convincing. In fact, it may be possible to devise games in which certain outcomes, according to the solution, make intuitively no sense. In an attempt to solve this problem, one may define a smaller set-valued solution concept that will reject more undesirable outcomes. This approach has been taken in the case of the bargaining set $M (= M_1^*)$ of Aumann and Maschler. In 1994, Granot (see also Granot and Maschler (1997)) introduced a new type of bargaining set, the reactive bargaining set. This new type is closely related to the original concept, however, it is a (proper) subset of the bargaining set. More recently, Sudhölter and Potters (2001) introduced the semireactive bargaining set which takes a middle position between the other two bargaining sets.

The underlying idea of each of these bargaining sets is that imputations which allow justified objections are rejected. As soon as an imputation is not a core allocation, certain coalitions have reasons to depart from the grand coalition and can raise an objection. If the objection is from player $k$ against player $\ell$, and the latter player is not able to counter, the objection is
justified. So, an objection can be interpreted as a complaint of player \( k \) against player \( \ell \). It consists of a coalition of players, say coalition \( P \), containing player \( k \) and not player \( \ell \), and an alternative solution for the players in this coalition which is strictly better than the initial solution. An interpretation is that player \( k \) tries to form a coalition in which he promises an improvement to get them on his side to object. Player \( \ell \) is able to counter this objection if he can find a coalition \( Q \) containing player \( \ell \) and not player \( k \) and an alternative proposal for the players in this coalition which is strictly better than the initial solution for the players outside coalition \( P \) (but members of coalition \( Q \)) and also weakly better than the alternative solution of player \( k \) for the players in \( P \) and \( Q \) (if there are any). An interpretation is that player \( \ell \) tries to find a coalition \( Q \) in which he can keep the players outside coalition \( P \), but in coalition \( Q \), content with respect to the initial solution and give the players in both coalitions \( P \) and \( Q \) at least the alternative solution offered by player \( k \).

So, these three bargaining sets have in common that they do not admit justified objections. They diverge with respect to the order in which the objections and counter-objections are announced. If in the bargaining set \( M \) player \( k \) plans to object against player \( \ell \), he has to announce the objection before player \( \ell \) has to state his counter-objection. So, player \( \ell \) knows completely the objection of player \( k \), before specifying his counter-objection. But in the reactive bargaining set \( M_r \), player \( \ell \) must choose a coalition for defending himself against every possible objection of player \( k \). So, before player \( k \) states his objection, player \( \ell \) should already specify the coalition he uses to counter. As this concept allows for more justified objections (player \( \ell \) has more difficulties to defend himself), the reactive bargaining set is a subset of the bargaining set. The semireactive bargaining set \( M_{sr} \) takes a middle position between these two concepts. In the semireactive bargaining set player \( k \) has only to announce in advance the coalition which he plans to object with. Player \( k \) may wait to specify the objection via this coalition, until player \( \ell \) has specified the coalition he uses to counter. So, player \( \ell \) only knows the coalition player \( k \) uses to object with, before he should give the coalition to counter. In view of how much player \( k \) is able to react to the coalition announced by player \( \ell \) and thus to have better possibilities for a justified objection, we have for a Transferable Utility (TU) game \( (N, v) \) the following inclusions

\[
M_r(v) \subseteq M_{sr}(v) \subseteq M(v).
\]

Sudhölter and Potters (2001) give examples of TU-games for which each of these inclusions are strict. Granot (1994) proves that \( M_r(v) \) is non-empty for all TU-games \( (N, v) \) with a non-empty imputation set. The proof of this statement boils down to the fact that the kernel (Davis and Maschler (1965)) is contained in the reactive bargaining set which, in turn, is proven to be non-empty by the same authors. Moreover, the reactive bargaining set contains the core whenever the latter is non-empty. An easy and straightforward argument is that imputations in the core do not allow any objections and therefore there is nothing to counter at all. By the inclusions mentioned above, both statements also hold for the semireactive bargaining set and the bargaining set. In general each of these bargaining sets may properly contain the core. However, in the literature several classes of TU-games can be found for which one of these three bargaining sets and the core coincide. Let us summarize these classes of TU-games.

The bargaining set and the core coincide for:
• Convex games (Maschler, Peleg and Shapley (1972)),
• Strongly balanced partitioning games, including e.g., assignment games and Γ-component additive games (Potters and Reijnierse (1995)),
• Non-negative superadditive games with a veto player (Potters, Muto and Tijs (1989)),
• Clan games (Potters, Poos, Tijs and Muto (1989)),
• Supperadditive simple flow games (Reijnierse, Maschler, Potters and Tijs (1996)),
• Three-person and four-person balanced games (Solymosi (2002)).

Of course, by the inclusions mentioned earlier, for these classes of TU-games the semireactive and reactive bargaining set also coincide with the core. Furthermore, the semireactive bargaining set (and thus also the reactive bargaining set) coincides with the core for:

• Superadditive simple games, including e.g., apex games and superadditive weighted majority games (Sudhölter and Potters (2001)).

Finally, the reactive bargaining set and the core coincide for:

• Simple network games (Granot (1994) and Granot, Granot and Zhu (1997)).

In this chapter we give, for each bargaining set, a new class of TU-games for which the particular bargaining set and the core coincide. We restrict our analysis to the class of symmetric TU-games. Of course, neither of the three bargaining sets can be seen as a real solution concept for symmetric TU-games, since it is obvious how the value of the grand coalition should be divided among the players. Nevertheless, already on this rather simple class of TU-games, these bargaining sets demonstrate a different behavior. That is, for each of the three bargaining sets different assumptions on the structure of the symmetric TU-game are needed in order for the particular bargaining set and the core to coincide. It turns out that for every symmetric TU-game the reactive bargaining set is the union of the core and the kernel. If the TU-game is superadditive, the same statement holds for the semireactive bargaining set. For the bargaining set and the core to coincide for a symmetric TU-game, total balancedness is sufficient. By examples we illustrate that each of these additional assumptions cannot be omitted. It is a straightforward observation that for a symmetric TU-game, the kernel is a singleton (i.e., the value of the grand coalition is equally divided among the players). So, for an arbitrary symmetric TU-game, the reactive bargaining set is the only concept among the three bargaining sets providing us with those solutions which one, intuitively speaking, would hope for. If the symmetric TU-game is not balanced, it gives ‘equal split’ as a solution and in case it is balanced, core allocations are provided as a solution (i.e., if there is an objection, then it is justified).

The chapter contains also a proof that for an arbitrary TU-game the bargaining set and the core coincide whenever the value of the grand coalition is ‘large enough’. This means that for a given incomplete TU-game (i.e., the value of the grand coalition is not yet specified) one can calculate a worth such that the bargaining set and the core coincide whenever the
value of the grand coalition exceeds this worth. A quite intriguing example of a symmetric TU-game, however, illustrates that the property ‘the bargaining set coincides with the core’ is not a prosperity property (Gellekom, Potters and Reijnierse (1999)). This means that for an incomplete TU-game there does in general not exist a worth such that the bargaining set and the core are the same sets if and only if the value of the grand coalition exceeds this worth.

2.2 Transferable utility games

In cooperative game theory it is assumed that the players’ first concern is to form a coalition in which they can profitably cooperate. Once the coalition has been formed, the players choose a joint action that generates an outcome. Each player has his own utility evaluation on the outcome, so the choice of a joint action and the generated outcome will therefore be the subject of negotiations. A well-known and extensively used model in cooperative game theory is the so-called finite n-person game with transferable utility (TU-game). This section is devoted to give a brief overview of the theory of TU-games. We recall some basic solution concepts along with some properties of these solution concepts. Also some properties on the structure of TU-games are given. For a more complete overview the reader is referred to Peleg and Sudhölter (2003). Let us start by giving the formal definition of a TU-game.

**Definition.** A **TU-game** is described by a pair \((N; v)\), where \(N\) is a finite set of players and where \(v : 2^N \rightarrow \mathbb{R}\) is a characteristic function which assigns to every coalition \(S \subseteq N\) a real number \(v(S)\), the value of coalition \(S\), with the convention that \(v(\emptyset) = 0\). The set \(N\) is called the grand coalition.

The value \(v(S) \in \mathbb{R}\) denotes the maximum profit that coalition \(S \subseteq N\) can generate by joint actions when it forms. Often it is assumed that the grand coalition \(N\) is formed, in other words all players of the TU-game cooperate. The term ‘transferable utility’ refers to the assumption that utility can be transferred from one player to another player and that the utilities of the players are comparable. The easiest situation which satisfies this assumption is when all values \(v(S)\) are expressed in terms of money, all players have enough money and they have the same linear utility for money. In Chapter 3 exchange economies with indivisible goods, money and quasi-linear utilities are studied from which such cooperative games arise in a natural way. Also the resource sharing problems studied in Chapter 5 and in Chapter 6 generate TU-games in an obvious way.

Once the model for a TU-game is given, the next step is to solve it. In this case, given that the grand coalition forms, this means to recommend fair allocations of the value \(v(N)\) among the players in \(N\). In the theory of cooperative TU-games many solution concepts can be found. A solution concept \(\varphi\) assigns to every TU-game \((N, v)\) (within some class of TU-games) a subset \(\varphi(N, v)\) of \(\mathbb{R}^N\). We usually write \(\varphi(v)\) instead of \(\varphi(N, v)\) as long as there is no doubt about the set of players \(N\). Before we recall the definitions of some basic and well-known solution concepts, we first give some desirable properties for a solution concept to satisfy.

**Definition.** A solution concept \(\varphi\) is:

- Efficient whenever \(x(N) = v(N)\) for all \(x \in \varphi(v)\) and every \((N, v)\),
2.2 Transferable utility games

- **Individually rational** whenever \( x_i \geq v(\{i\}) \) for all \( i \in N \), all \( x \in \varphi(v) \) and every \( \langle N, v \rangle \),
- **Coalition rational** whenever \( x(S) \geq v(S) \) for all \( S \subseteq N \), all \( x \in \varphi(v) \) and every \( \langle N, v \rangle \).

For all \( S \subseteq N \) we use the standard shorthand notation \( x(S) := \sum_{i \in S} x_i \).

Efficiency states that the value \( v(N) \) is completely allocated among the players. Individual rationality states that each player receives at least the profit that he can achieve by himself, whereas coalition rationality states that each coalition receives at least the maximum profit that they can obtain by themselves.

**Definition.** The **imputation set** \( I(v) \) of a TU-game \( \langle N, v \rangle \) contains all allocations which are efficient and individually rational, i.e.,

\[
I(v) := \{ x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N \}.
\]

Clearly, the imputation set is non-empty if and only if \( \sum_{i \in N} v(\{i\}) \leq v(N) \). Throughout the chapter, we assume that \( I(v) \neq \emptyset \). For a TU-game with a non-empty imputation set, this set is often very large and therefore this set is hardly seen as a solution concept, but as a desirable set in which every solution concept should be contained. An example of such a solution concept is the **core**.

**Definition (Gillies (1959)).** The **core**, denoted by \( C(v) \), of a TU-game \( \langle N, v \rangle \) contains all allocations which are efficient and coalition rational, i.e.,

\[
C(v) := \{ x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N \}.
\]

So, the core contains all imputations for which each coalition receives at least the maximal profit its members can generate by joint actions. This set of solutions is stable in the sense that it gives no coalition an incentive to split off. However, the core may be empty.

**Example.** Let \( \langle N, v \rangle \) be a TU-game with \( N = \{1, 2, 3\} \) and the values \( v(S) \) for \( S \subseteq N \) given by:

\[
\begin{align*}
v(N) &= 12, \\
v(\{1, 2\}) &= 10, \quad v(\{1, 3\}) = 8, \quad v(\{2, 3\}) = 7, \\
v(\{1\}) &= 1, \quad v(\{2\}) = 3 \text{ and } v(\{3\}) = 1.
\end{align*}
\]

Observe that \( I(v) \neq \emptyset \), since \( v(\{1\}) + v(\{2\}) + v(\{3\}) = 1 + 3 + 1 = 5 < 12 = v(N) \). Suppose \( C(v) \neq \emptyset \) and let \( x \in C(v) \). Then \( x \in \mathbb{R}^3 \) satisfies the following inequalities:

\[
\begin{align*}
x_1 + x_2 + x_3 &= 12, \\
x_1 + x_2 &\geq 10, \\
x_1 + x_3 &\geq 8, \\
x_2 + x_3 &\geq 7, \\
x_1 &\geq 1, \quad x_2 \geq 3, \quad x_3 \geq 1.
\end{align*}
\]

Hence,

\[
24 = 2 \cdot (x_1 + x_2 + x_3) = (x_1 + x_2) + (x_1 + x_3) + (x_2 + x_3) \geq 10 + 8 + 7 = 25.
\]

This contradiction shows that the core \( C(v) \) is empty.
Next, we recall a well-known theorem, proven independently by Bondareva (1963) and Shapley (1967), which provides a necessary and sufficient condition for the core to be non-empty. For that purpose, we first need the definition of balancedness. Given a coalition $S \subseteq N$ its characteristic vector $e_S \in \mathbb{R}^N$ is defined by $(e_S)_i = 1$ if $i \in S$ and $(e_S)_i = 0$ else.

**Definition.** A TU-game $\langle N, v \rangle$ is balanced if for every non-negative solution $\{A_S\}_{S \subseteq N}$ of the equation $\sum_{S \subseteq N} A_S \cdot e_S = e_N$ the inequality $\sum_{S \subseteq N} A_S \cdot v(S) \leq v(N)$ holds.

**Theorem 2.1 (Bondareva (1963), Shapley (1967)).** Let $\langle N, v \rangle$ be a TU-game. Then the core $C(v)$ is non-empty if and only if $\langle N, v \rangle$ is balanced.

So, if a TU-game is balanced, then the core is non-empty and therefore it is a convenient solution concept on the class of balanced TU-games. But what if the core is empty? Assume we have an imputation $x \in I(v)$ and the TU-game $\langle N, v \rangle$ is not balanced. Then there exists a coalition $S \subseteq N$ which can depart from the grand coalition $N$ and improve with respect to the imputation $x \in I(v)$. However, this coalition $S$ still faces a problem, namely, how to divide the value $v(S)$ among the players in $S$. Although all the players in $S$ can improve, with respect to this imputation $x \in I(v)$, there may be reallocations of the value $v(S)$ which trigger alternative proposals of players outside $S$ in such a way that maybe some players in $S$ do not want to split off. Aumann and Maschler (1964) proposed a solution concept which takes this issue into account. Their idea is that for a coalition to split off, it must be justified by the absence of another coalition which can give an argument for the coalition not to split off. In the next section we give, next to the approach by Aumann and Maschler, two more recent approaches to formalize this idea.

This section we end by recalling some well-known properties on the structure of a TU-game and we recall the definition of a prosperity property. Let us start by giving the definition of superadditivity.

**Definition.** A TU-game $\langle N, v \rangle$ is superadditive if $v(S) + v(T) \leq v(S \cup T)$ whenever $S, T \subseteq N$ and $S \cap T = \emptyset$.

In most of the applications of TU-games superadditivity is satisfied. Indeed, it may be argued that if coalition $S \cup T$ forms, its members can decide to act as if $S$ and $T$ would have formed separately and by doing so they generate $v(S) + v(T)$. Nevertheless, superadditivity may be violated. The members of the coalitions $S$ and $T$ may hamper each other or they find it more difficult to reach an agreement in a larger coalition. An even more demanding property for a TU-game is total balancedness. For all $S \subseteq N$ we write $v_S : 2^S \to \mathbb{R}$ as the restriction of $v$ to coalition $S$.

**Definition.** A TU-game $\langle N, v \rangle$ is totally balanced if for every coalition $S \subseteq N$ the subgame $\langle S, v_S \rangle$ is balanced.

Total balancedness states that every subgame has a non-empty core. Hence, a totally balanced TU-game is in particular balanced. It is also straightforward that total balancedness implies superadditivity, however, there are superadditive TU-games which are not balanced and therefore not totally balanced. See for instance the example of the three-person TU-game, given earlier in this section, which is superadditive but not (totally) balanced. Next, we mention the concept of convexity.
2.2 Transferable utility games

Definition (Shapley (1971)). A TU-game \((N, v)\) is convex if
\[
v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad \text{for all } S, T \subseteq N.
\]
Convexity implies total balancedness (Shapley (1971)) and thus a convex TU-game is in particular balanced.

Next, we give four additional properties on a TU-game. Let us start by giving the definition of extendability which states that any core element of any subgame can be extended to a core element of \((N, v)\).

Definition (Kikuta and Shapley (1986)). Let \((N, v)\) be a TU-game. The core \(C(v)\) is extendable if for every coalition \(S \subseteq N\) and every core element \(y\) of the subgame \((S, v_S)\) there exists an allocation \(x \in C(v)\) such that \(x_i = y_i\) for all \(i \in S\).

Two other properties on a TU-game that can be found in the literature are largeness and exactness.

Definition (Sharkey (1982)). Let \((N, v)\) be a TU-game. The core \(C(v)\) is large if for every vector \(y \in \mathbb{R}^N\) with \(y(S) \geq v(S)\) for all \(S \subseteq N\) there exists an allocation \(x \in C(v)\) such that \(x_i \leq y_i\) for all \(i \in N\).

Definition (Schmeidler (1972)). Let \((N, v)\) be a TU-game. The core \(C(v)\) is exact if for every coalition \(S \subseteq N\) there exists an allocation \(x \in C(v)\) such that \(x(S) = v(S)\).

Finally, we mention the concept of stability of the core. The core is stable whenever every imputation outside the core is dominated by an element of the core.

Definition. Let \((N, v)\) be a TU-game. The core \(C(v)\) is stable if for every imputation \(y \in \mathcal{I}(v) \setminus C(v)\) there exists an allocation \(x \in C(v)\) and a coalition \(S \subseteq N\) which dominates \(y\) via \(S\), i.e., \(x(S) = v(S)\) and \(x_i > y_i\) for all \(i \in S\). If the core is stable then it is the unique Von Neumann-Morgenstern solution (1944).

The following relations between extendability, largeness and stability of the core can be found in Kikuta and Shapley (1986) and are stated here without a proof.

Theorem 2.2 (Kikuta and Shapley (1986)). If \((N, v)\) is a TU-game with a large core \(C(v)\), then \(C(v)\) is extendable. If \((N, v)\) is a TU-game with an extendable core \(C(v)\), then \(C(v)\) is stable.

In fact, extendability is the weakest sufficient condition known at the moment in order for a TU-game to have a stable core. Gellekom et al. (1999) give an example which shows that extendability and stability of the core are no equivalent properties. Furthermore, they prove that the properties largeness and non-emptiness of the core are so-called prosperity properties. This means that if we start with an arbitrary TU-game and increase only the value \(v(N)\) of the grand coalition, these properties will arise at a given moment and will be kept if we go on with increasing this value. Let us give the formal definition of this concept. The restriction of \(v\) to \(2^N \setminus \{N\}\) is denoted by \(v^0\) and \((N, v^0)\) is called an incomplete TU-game. Using this notation, a TU-game \((N, v)\) in which \(v(N) = \gamma\) for a certain \(\gamma \in \mathbb{R}\) can also be denoted by \((N, v^0, v(N) = \gamma)\).
Definition (Gellekom et al. (1999)). A property $P$ on TU-games is a prosperity property if for every incomplete TU-game $(N, v^0)$ there exists a number $\alpha_P(v^0) \in \mathbb{R}$ such that

The TU-game $(N, v^0, v(N))$ has property $P$ if and only if $v(N) \geq \alpha_P(v^0)$.

A property $P$ on TU-games is a weak prosperity property if for every incomplete TU-game $(N, v^0)$ there exists a number $\alpha_P(v^0) \in \mathbb{R}$ such that

- The TU-game $(N, v^0, v(N))$ has property $P$ if $v(N) > \alpha_P(v^0)$,
- For every $\beta < \alpha_P(v^0)$ there exists a number $\gamma \in (\beta, \alpha_P(v^0))$ such that the TU-game $(N, v^0, v(N) = \gamma)$ does not have property $P$.

So, the difference between a prosperity property and a weak prosperity property on a TU-game $(N, v^0, v(N))$ is that in the latter definition the property $P$ may either hold or not hold whenever $v(N) \leq \alpha_P(v^0)$. Gellekom et al. (1999) also introduced the concept of a monotone and closed property on TU-games which closes the gap between a weak prosperity property and a prosperity property.

Definition. A property $P$ on TU-games is monotone if for every incomplete TU-game $(N, v^0)$ and every $\gamma \geq \beta$ we have,

$(N, v^0, v(N) = \beta)$ has property $P \Rightarrow (N, v^0, v(N) = \gamma)$ has property $P$.

A property $P$ on TU-games is closed if for every incomplete TU-game $(N, v^0)$ and every sequence $\beta_m \downarrow \beta$ we have,

$(N, v^0, v(N) = \beta_m)$ has property $P \Rightarrow (N, v^0, v(N) = \beta)$ has property $P$.

Remark. The following two statements follow immediately from the definitions.

(i) If $P$ is a weak prosperity property and in addition monotone and closed, then $P$ is a prosperity property.

(ii) If a (weak) prosperity property $P$ implies property $P'$, then $P'$ is a weak prosperity property.

An obvious example of a prosperity property is non-emptiness of the imputation set. Indeed, given an incomplete TU-game $(N, v^0)$, the TU-game $(N, v^0, v(N))$ has a non-empty imputation set if and only if $v(N) \geq \alpha_T(v^0)$, where $\alpha_T(v^0) := \sum_{i \in N} v^0(i)$. An other example of a prosperity property is balancedness, since the condition for a TU-game $(N, v)$ to be balanced can, according to Theorem 2.1, be written as,

$$v(N) \geq \max \{ \sum_{S \subseteq N} \lambda_S \cdot v^0(S) \mid \lambda_S \geq 0 \text{ such that } \sum_{S \subseteq N} \lambda_S \cdot e_S = e_N \}.$$

In fact, given an incomplete TU-game $(N, v^0)$, the value $\alpha_{Bal}(v^0)$ can be computed by solving a linear program. Observe that $\alpha_{Bal}(v^0) \geq \alpha_T(v^0)$ for every incomplete TU-game $(N, v^0)$. Total balancedness and superadditivity are no (weak) prosperity properties,
since violated inequalities not concerning the grand coalition \(N\) will remain violated, even if one increases the value \(v(N)\). Gellekom et al. (1999) prove that largeness of the core and extendability are prosperity properties. Note that according to Theorem 2.2 it follows that \(\alpha_{LC}(v^0) \geq \alpha_{Ext}(v^0)\) for every incomplete TU-game \(\langle N, v^0 \rangle\). Exactness is not a (weak) prosperity property, since exactness implies total balancedness. Stability of the core is a weak prosperity property, since extendability implies stability. It is still an open problem whether stability of the core is a closed and/or monotone property.

In Section 2.4 we prove that the property ‘the bargaining set and the core coincide’ is a weak prosperity property. However, later on in this chapter, in Section 2.6, we give an example which illustrates that this property is neither monotone nor closed and therefore it is not a prosperity property. In this example the bargaining set and core coincide for all possible values of \(v(N)\), larger or equal than \(Bal(v^0)\), except for exactly one value for \(v(N)\). For this value the bargaining set is equal to the union of the core with a finite set of imputations outside the core.

\section*{2.3 Bargaining sets in TU-games}

In this part of the chapter three solution concepts for TU-games are considered which can be seen as a substitute of the core in the case the latter is empty. Each of these solution concepts are based on the idea that the players have certain bargaining possibilities. Before we can recall the formal definitions of these three concepts, we first need the notion of an objection. For all players \(k, \ell \in N\) with \(k \neq \ell\) we write \(k\ell := \{S \subseteq N \mid k \in S \subseteq N \setminus \{\ell\}\}\) for the set of coalitions containing player \(k\) but not player \(\ell\).

**Definition.** An objection of player \(k\) against player \(\ell\) with respect to an imputation \(x \in I(v)\) is a pair \((P, y)\) with \(P \in \Gamma_{k\ell}\) and \(y \in \mathbb{R}^P\) such that

\[
y_i > x_i \quad \text{for all } i \in P \quad \text{and} \quad y(P) \leq v(P).
\]

So, an objection \((P, y)\) is an incentive for the players in coalition \(P\) to withdraw from the grand coalition \(N\). That is, each of the players in \(P\) has a reason to complain (against player \(\ell\)), since coalition \(P\) can divide the value \(v(P)\) among themselves in such a way that all members improve with respect to the imputation \(x \in I(v)\). But perhaps player \(\ell\) is able to give a counter-objection to each complaint which makes it not justified. The existence of such a counter-objection is the key-concept in the idea of the bargaining sets. Without any further delay we now give the definitions of the bargaining set, the semireactive and the reactive bargaining set of a TU-game.

**Definition (Aumann and Maschler (1964)).** Let \(\langle N, v \rangle\) be a TU-game. An imputation \(x \in I(v)\) is an element of the bargaining set \(\mathcal{M}(v)\) if for all players \(k\) and \(\ell\) in \(N\) and for every objection \((P, y)\) of \(k\) against \(\ell\) with respect to \(x\) there exists a coalition \(Q \in \Gamma_{k\ell}\) such that

\[
y(P \cap Q) + x(Q \setminus P) \leq v(Q).
\]

**Definition (Sudhölter and Potters (2001)).** Let \(\langle N, v \rangle\) be a TU-game. An imputation \(x \in I(v)\) is an element of the semireactive bargaining set \(\mathcal{M}_{sr}(v)\) if for all players \(k\) and \(\ell\) in \(N\)
and for every coalition \( P \in \Gamma_{k\ell} \) there exists a coalition \( Q \in \Gamma_{k\ell} \) such that for every objection \( (P, y) \) of \( k \) against \( \ell \) with respect to \( x \), it holds that \( y(P \cap Q) + x(Q \setminus P) \leq v(Q) \).

**Definition (Granot (1994)).** Let \( (N, v) \) be a TU-game. An imputation \( x \in \mathcal{I}(v) \) is an element of the **reactive bargaining set** \( \mathcal{M}_r(v) \) if for all players \( k \) and \( \ell \) in \( N \) there exists a coalition \( Q \in \Gamma_{k\ell} \) such that for every objection \( (P, y) \) of \( k \) against \( \ell \) with respect to \( x \), it holds that \( y(P \cap Q) + x(Q \setminus P) \leq v(Q) \).

**Remark.** Let \( (P, y) \) be an objection of player \( k \) against player \( \ell \) with respect to \( x \in \mathcal{I}(v) \) such that \( y(P \cap Q) + x(Q \setminus P) \geq v(Q) \) for every \( Q \in \Gamma_{k\ell} \). Then for every \( Q \in \Gamma_{k\ell} \) there does not exist a vector \( z \in \mathbb{R}^Q \) with \( z(Q) = v(Q) \) such that
\[
z_i \geq y_i \quad \text{for all } i \in P \cap Q \quad \text{and} \quad z_i \geq x_i \quad \text{for all } i \in Q \setminus P.
\]

This means that player \( \ell \) does not have a **counter-objection** \( (Q, z) \) against the objection of player \( k \). In this case the objection \( (P, y) \) is said to be **justified**.

In each of the mentioned bargaining sets player \( \ell \) should be able to counter every possible objection of player \( k \). The difference between these three bargaining sets is the amount of information player \( \ell \) has about the objection of player \( k \) before he has to specify the coalition \( Q \in \Gamma_{k\ell} \) he uses to counter. In the bargaining set player \( \ell \) **completely** knows how the objection \( (P, y) \) of player \( k \) looks like. In the semireactive bargaining set player \( \ell \) **at least** knows the coalition \( P \in \Gamma_{k\ell} \) player \( k \) uses in his objection, but in the reactive bargaining set player \( \ell \) does not know anything about the possible objection from player \( k \). In view of how much information player \( \ell \) has about the objection of player \( k \) before he specifies the coalition \( Q \in \Gamma_{k\ell} \) he uses to counter, we have the following inclusions,
\[
\mathcal{M}_r(v) \subseteq \mathcal{M}_{sr}(v) \subseteq \mathcal{M}(v).
\]

The core \( \mathcal{C}(v) \) of a TU-game \( (N, v) \) is contained in the reactive bargaining set because of the obvious reason that if \( x \in \mathcal{C}(v) \) no player can raise an objection with respect to this allocation \( x \). So, on the class of balanced TU-games these three bargaining sets are non-empty. In search for the non-emptiness of the bargaining set of an arbitrary TU-game another solution concept came up in the literature, namely, the **kernel**.

**Definition (Davis and Maschler (1965)).** Let \( (N, v) \) be a TU-game. The **kernel** is defined by
\[
\mathcal{K}(v) := \{ x \in \mathcal{I}(v) \mid s_{k\ell}(x) \leq s_{k}(x) \text{ or } x_{\ell} = v(\{\ell\}) \text{ for all } k \neq \ell \in N \},
\]
where \( s_{k\ell}(x) := \max\{v(S) - x(S) \mid S \in \Gamma_{k\ell} \} \) for all \( k, \ell \in N \) with \( k \neq \ell \) denotes the **maximum surplus** of player \( k \) against player \( \ell \).

Davis and Maschler (1965) prove that the kernel is a subset of the bargaining set and that it is non-empty (given that the imputation set is non-empty). As a result, non-emptiness of the bargaining set is guaranteed. This result can be generalized to the reactive bargaining set. This is demonstrated in Granot (1994). For the sake of completeness we give also a proof.

**Theorem 2.3 (Granot (1994)).** Let \( (N, v) \) be a TU-game, then \( \mathcal{K}(v) \subseteq \mathcal{M}_r(v) \).
2.3 Bargaining sets in TU-games

Proof. Let \( (N, v) \) be a TU-game and take \( x \in K(v) \). Let \( k \in N \) be a player who is able to raise an objection against player \( \ell \) with respect to \( x \). Observe that this in particular implies that \( s_{k\ell}(x) > 0 \). In case \( x_{\ell} = v(\{\ell\}) \), player \( \ell \) is able to counter every objection of player \( k \) via the one-coalition \( \{\ell\} \). On the other hand, if \( x_{\ell} > v(\{\ell\}) \), then player \( \ell \) needs an other coalition to counter. Because \( x \in K(v) \) we obtain in this case that \( 0 < s_{k\ell}(x) \leq s_{k\ell}(x) \).

Take \( Q \in \Gamma_{k\ell} \) such that \( v(Q) - x(Q) = s_{k\ell}(x) \). Suppose there exists an objection \( (P, y) \) of player \( k \) against player \( \ell \) which cannot be countered by coalition \( Q \), i.e.,

\[
y(P \cap Q) + x(Q \setminus P) > v(Q).
\]

Then

\[
v(P) - x(P) \geq y(P) - x(P) = y(P \cap Q) - x(P \cap Q) + y(P \setminus Q) - x(P \setminus Q) > y(P \cap Q) - x(P \cap Q) = y(P \cap Q) + x(Q \setminus P) - x(Q) > v(Q) - x(Q).
\]

Therefore, \( s_{k\ell}(x) \geq v(P) - x(P) > v(Q) - x(Q) = s_{k\ell}(x) \). However, \( x_{\ell} > v(\{\ell\}) \) and \( x \in K(v) \). Contradiction. Hence, player \( \ell \) is able to counter the objection \( (P, y) \) via coalition \( Q \).

The definition of the reactive bargaining set leaves the possibility open for one more refinement of the bargaining set of Aumann and Maschler, in terms of the amount of information the players have about possible objections. Assume player \( \ell \) can expect an objection of player \( k \). To counter this objection, player \( \ell \) should not only specify in advance the coalition \( Q \in \Gamma_{k\ell} \) he uses to counter, but he also should give the alternative solution \( z \in \mathbb{R}^Q \) for the players in \( Q \). After player \( \ell \) has given this pair \((Q, z)\) player \( k \) may choose his objection \((P, y)\) against player \( \ell \). Clearly, this approach leads to a refinement of the reactive bargaining set, since this new concept allows for more justified objections (player \( \ell \) has more difficulties to defend himself). However, for this refinement non-emptiness cannot be guaranteed.

Example. Let \( (N, v) \) be a simple, superadditive and non-balanced TU-game with \( N = \{1, \ldots, 4\} \) and the values \( v(S) \) for \( S \subseteq N \) given by:

\[
\begin{align*}
v(N) &= 1 \\
v(\{1, 2, 3\}) &= v(\{1, 2, 4\}) = v(\{1, 3, 4\}) = v(\{2, 3, 4\}) = 1 \\
v(\{1, 2\}) &= v(\{1, 3\}) = 1 \\
v(S) &= 0 \text{ else.}
\end{align*}
\]

Some straightforward calculations yields that the kernel \( K(v) \) of this TU-game is a singleton, namely, \( K(v) = \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)\} \). Granot et al. (1997) show that the reactive bargaining set of a simple, superadditive and non-balanced TU-game coincides with the kernel. Therefore, the refinement of the reactive bargaining set in which the objector \( k \) may choose his objection after player \( \ell \) has given his counter-objection should contain \( x := (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) \). Otherwise, this refinement is empty.

However, player 1 can make an objection against player 2 with respect \( x \). The only possible coalition player 2 can use to counter is \( Q := \{2, 3, 4\} \). Player 1 can make the objection
Bargaining Sets in Cooperative Games

the objection \((f, 3, g, (5, 12, 12, 2, 12))\). If player 2 would be able to counter both objections by one counter-objection \((Q; z)\) for a certain \(z \in \mathbb{R}^Q\), then \(z_2 \geq \frac{1}{3}\), \(z_3 \geq \max\{\frac{1}{5}, \frac{1}{12}\} = \frac{1}{5}\) and \(z_4 \geq \frac{1}{12}\). But then the total payoff according to this vector \(z \in \mathbb{R}^Q\) exceeds the value of \(Q\), since

\[ z(\{2, 3, 4\}) \geq \frac{1}{5} > 1 = v(\{2, 3, 4\}). \]

Hence, for every counter-objection \((Q, z)\) of player 2 there exists an objection of player 1 which cannot be countered by \((Q, z)\).

The example illustrates that in search for refinements of the bargaining set of Aumann and Maschler, the reactive bargaining set is the furthest one can go, still guaranteeing non-emptiness. Differently, to maintain non-emptiness, which is game theoretically speaking important, one cannot ask from a player under attack to give more information in advance than the coalition he uses to counter.

2.4 Bargaining sets and the core

According to Theorem 2.3 each of the three bargaining sets, studied in this chapter, are non-empty (given of course that the imputation set is non-empty). Hence, when the core is non-empty each bargaining set contains the core, but in contrast with the core, however, non-emptiness can be guaranteed for each bargaining set. Therefore, one can see the three mentioned bargaining sets as substitutes of the core whenever it is empty. This gives rise to the question: ‘When is one of these bargaining sets a real extension of the core?’ To put it differently: ‘For which balanced TU-games does the core coincide with one of the bargaining sets?’

In Section 2.5 this question is studied for symmetric TU-games. In the present section, we prove that the property ‘the bargaining set coincides with the core’ is a weak prosperity property. Intuitively, this means that the bargaining set and the core coincide whenever the value of the grand coalition is large enough. Observe that this also yields the equivalence of the semireactive and reactive bargaining set with the core whenever the value of the grand coalition is large enough. To prove this statement, we first need some preparations.

Given an incomplete TU-game \(\langle N, v^0 \rangle\) (see Section 2.2 for the definition), we define for all \(i \in N\),

\[ \mathcal{A}_i^+(v^0) := \{ x \in \mathbb{R}^N \mid x(S) > v^0(S) \text{ for all } S \neq N \text{ with } i \in S \}. \]

And given a TU-game \(\langle N, v \rangle\), we denote

\[ \mathcal{A}_i(v) := I(v) \cap \mathcal{A}_i^+(v^0) \text{ for all } i \in N. \]

Given these notational conventions, we now can present a sufficient condition for the bargaining set and the core, of an arbitrary balanced TU-game, to coincide.

**Lemma 2.4.** Let \(\langle N, v \rangle\) be a balanced TU-game such that \(I(v) = \bigcup_{i \in N} \mathcal{A}_i(v),\) then

\[ M(v) = C(v). \]
2.4 Bargaining sets and the core

**Proof.** Let \( (N, v) \) be a balanced TU-game such that \( \mathcal{I}(v) = \bigcup_{i \in N} \mathcal{A}_i(v) \) and let \( x \in \mathcal{I}(v) \) be an imputation outside the core \( C(v) \). Then there exists \( \ell \in N \) such that \( x \in \mathcal{A}_\ell(v) \). Because \( x \notin C(v) \), there exists a coalition \( P \subseteq N \) such that \( v(P) > x(P) \). Observe that \( \ell \notin P \).

Take \( k \in P \). We define the objection \( (P, y) \) of player \( k \) against player \( \ell \) with respect to \( x \) by

\[
y_i := x_i + \frac{1}{|P|} [v(P) - x(P)] \quad \text{for all } i \in P.
\]

Take \( Q \in \Gamma_{ik} \), then it follows that

\[
y(P \cap Q) + x(Q \setminus P) \geq x(Q) > v(Q).
\]

The last inequality follows from the fact that \( x \in \mathcal{A}_\ell(v) \). Hence, the objection \( (P, y) \) is justified.

Next, we prove that the property \( \mathcal{I}(v) = \bigcup_{i \in N} \mathcal{A}_i(v) \) is a monotone and weak prosperity property.

**Proposition 2.5.** Let \( (N, v^0) \) be an incomplete TU-game. There exists a value \( \alpha_{\mathcal{A}}(v^0) \geq \alpha_{\mathcal{I}}(v^0) \) such that

\[
\mathcal{I}(v^0, v(N) = \beta) = \bigcup_{i \in N} \mathcal{A}_i(v^0, v(N) = \beta) \quad \text{if and only if } \beta > \alpha_{\mathcal{A}}(v^0).
\]

**Proof.** Let \( (N, v^0) \) be an incomplete TU-game. Without loss of generality, we assume that \( v^0([i]) = 0 \) for all \( i \in N \). Define,

\[
B(v^0) := \{ \beta \geq \alpha_{\mathcal{I}}(v^0) | \mathcal{I}(v^0, v(N) = \beta) = \bigcup_{i \in N} \mathcal{A}_i(v^0, v(N) = \beta) \}.
\]

**Claim 1.** \( B(v^0) \neq \emptyset \).

**Proof.** Define \( \alpha_i := \max_{S \subseteq N} v^0(S) \) for all \( i \in N \). Take \( \beta > \sum_{i \in N} \alpha_i \) and let \( x \in \mathcal{I}(v^0, v(N) = \beta) \). Then, due to efficiency, there exists a player \( i \in N \) such that \( x_i > \alpha_i \). Hence, since \( x_j \geq v^0([j]) = 0 \) for all \( j \in N \), we have for every \( S \neq N \) with \( i \in S \) that

\[
x(S) \geq x_i > \alpha_i \geq v^0(S).
\]

So, we can conclude that \( x \in \mathcal{A}_i(v^0, v(N) = \beta) \). Hence, \( B(v^0) \neq \emptyset \).

**Claim 2.** If \( \beta \in B(v^0) \) and \( \gamma \geq \beta \), then \( \gamma \in B(v^0) \).

**Proof.** Let \( \gamma \geq \beta \) for a certain \( \beta \in B(v^0) \). Take \( y \in \mathcal{I}(v^0, v(N) = \gamma) \). Then it can be verified that there exists \( x \in \mathcal{I}(v^0, v(N) = \beta) \) such that \( x_j \leq y_j \) for all \( j \in N \). Because \( \beta \in B(v^0) \), we know that there exists a player \( i \in N \) such that \( x(S) > v^0(S) \) for all \( S \neq N \) with \( i \in S \). Hence, \( y \in \mathcal{A}_i(v^0, v(N) = \gamma) \) and thus \( \gamma \in B(v^0) \).

Given these two statements on \( B(v^0) \), we now define,

\[
\alpha_{\mathcal{A}}(v^0) := \inf_{\beta \in B(v^0)} \beta.
\]
By definition of $B(v^{0})$ and due to Claim 1, $\alpha_{A}(v^{0})$ is well-defined. Claim 2 tells us that

$$\alpha_{A}(v^{0}), \infty \subseteq B(v^{0}) \subseteq [\alpha_{A}(v^{0}), \infty).$$

So, it is left to prove that $\alpha_{A}(v^{0}) \notin B(v^{0})$.

To obtain a contradiction, suppose that $\alpha_{A}(v^{0}) \in B(v^{0})$. For simplicity, we denote $\alpha_{A} := \alpha_{A}(v^{0})$ and use the abbreviation $I(\alpha_{A})$ for the imputation set $I(v^{0}, v(N) = \alpha_{A}(v^{0}))$. Then for every $x \in I(\alpha_{A})$ there exists a player $i \in N$ such that

$$x \in A_{i}(v^{0}, v(N) = \alpha_{A}) = I(\alpha_{A}) \cap A^{*}_{i}(v^{0}).$$

Because $A^{*}_{i}(v^{0})$ is an open set in $\mathbb{R}^{N}$, there exists for every $x \in I(\alpha_{A})$ a number $\varepsilon(x) > 0$ such that

$$x \in B_{\frac{1}{2}\varepsilon(x)}(x) \subseteq B_{\varepsilon(x)}(x) \subseteq A^{*}_{i}(v^{0}).$$

Here, $B_{\varepsilon}(x) := \{y \in \mathbb{R}^{N} \mid \max_{i \in N} |y_{i} - x_{i}| < \varepsilon\}$ for all $x \in I(\alpha_{A})$ and every $\varepsilon > 0$.

The collection $\{B_{\frac{1}{2}\varepsilon(x)}(x)\}_{x \in I(\alpha_{A})}$ covers $I(\alpha_{A})$. Hence, due to compactness of the imputation set $I(\alpha_{A})$ we can conclude that

$$I(\alpha_{A}) \subseteq \bigcup_{k=1}^{K} B_{\frac{1}{2}\varepsilon(x^{k})}(x^{k}) \subseteq \bigcup_{k=1}^{K} B_{\varepsilon(x^{k})}(x^{k})$$

for certain $x^{1}, \ldots, x^{K} \in I(\alpha_{A})$. Define, $\varepsilon := \min\{\varepsilon(x^{k}) \mid 1 \leq k \leq K\}$ and select $\beta > 0$ such that $\alpha_{A} - \varepsilon < \beta < \alpha_{A}$. We show that $\beta$ is also in $B(v^{0})$, which yields the desired contradiction.

Take $y \in I(v^{0}, v(N) = \beta)$. Note that $y + \frac{\alpha_{A} - \beta}{|N|} e_{N} \in I(\alpha_{A})$. Select $1 \leq k \leq K$ such that

$$y + \frac{\alpha_{A} - \beta}{|N|} e_{N} \in B_{\frac{1}{2}\varepsilon(x^{k})}(x^{k}).$$

Then $y \in B_{\frac{1}{2}\varepsilon(x^{k})}(x^{k})$. Indeed, for all $j \in N$ we have that

$$|y_{j} - x^{k}_{j}| < \frac{1}{2}\varepsilon(x^{k}) + \frac{\alpha_{A} - \beta}{|N|} < \frac{1}{2}\varepsilon(x^{k}) + \frac{1}{2}\varepsilon \leq \varepsilon(x^{k}).$$

Hence, $y \in A^{*}_{i}(v^{0})$ and thus $y \in A_{i}(v^{0}, v(N) = \beta)$. From this we can conclude that $\beta \in B(v^{0})$. This contradicts the definition of $\alpha_{A}$, since $\beta < \alpha_{A}$. Hence, $B(v^{0}) = (\alpha_{A}(v^{0}), \infty)$. This completes the proof.

If we define $\alpha_{MC}(v^{0}) := \max\{\alpha_{Bet}(v^{0}), \alpha_{A}(v^{0})\}$ for every incomplete TU-game $(N, v^{0})$, we obtain according to Lemma 2.4 and Proposition 2.5 the following result.

**Corollary 2.6.** Let $(N, v^{0})$ be an incomplete TU-game, then

$$\mathcal{M}(v^{0}, v(N) = \beta) = \mathcal{C}(v^{0}, v(N) = \beta)$$

whenever $\beta > \alpha_{MC}(v^{0})$.

So, according to Corollary 2.6, the property ‘the bargaining set and the core coincide’ is a weak prosperity property. However, in Section 2.6 we give an example which illustrates that this property on a TU-game is neither closed nor monotone and thus it is not a prosperity property. In Section 2.6, we also compare for certain incomplete TU-games $(N, v^{0})$ the value $\alpha_{MC}(v^{0})$ with the values $\alpha_{Ext}(v^{0})$ and $\alpha_{LC}(v^{0})$ of largeness and extendability, respectively.
2.5 Symmetric TU-games

This section studies the three bargaining sets, defined in Section 2.3, on the class of symmetric TU-games. Let us start by recalling the definition of a symmetric TU-game.

A TU-game \( (N, v) \) is **symmetric** if there exists a map \( f : \{0, 1, \ldots, n\} \rightarrow \mathbb{R} \) with \( f(0) = 0 \) and \( v(S) = f(s) \) whenever \( s = |S| \) for all \( S \subseteq N \). So, in a symmetric TU-game the value of a coalition only depends on its size and not on its members. Since the characteristic function \( v : 2^N \rightarrow \mathbb{R} \) can be represented by the map \( f : \{0,1,\ldots,n\} \rightarrow \mathbb{R} \), we write \( h_{N; f} \) instead of \( h_{N; v} \).

A symmetric TU-game \( h_{N; f} \) is **balanced** if and only if \( f(s) = s \cdot f(n) \) for all \( 1 \leq s \leq n \).

It is **totally balanced** if and only if the map \( s \mapsto f(s) = s \cdot f(n) \) is weakly monotonic and it is **superadditive** if and only if \( f(s) + f(t) \leq f(s+t) \) for all \( 1 \leq s, t \leq n \). Recall from Section 2.2 that every totally balanced (symmetric) TU-game is in particular superadditive.

If \( h_{N; f} \) is a symmetric game and \( x \in I(f) \) an imputation, we assume, without loss of generality, throughout the remaining sections of the chapter that \( N := \{1, \ldots, n\} \) such that \( x_1 \leq x_2 \leq \ldots \leq x_n \).

### 2.5.1 The reactive and the semireactive bargaining set

This subsection studies the reactive and semireactive bargaining set for symmetric TU-games. Let us start by proving that on this class of TU-games the reactive bargaining set is the union of the core and the kernel.

**Theorem 2.8.** Let \( h_{N; f} \) be a symmetric TU-game, then \( M_r(f) = C(f) \cup K(f) \) if \( C(f) \neq \emptyset \) and \( K(f) \) if \( C(f) = \emptyset \).

**Proof.** Let \( h_{N; f} \) be a symmetric TU-game. Take \( x \in M_r(f) \). Suppose \( x \notin C(f) \cup K(f) \). So, according to Lemma 2.7, this yields that \( x_n > x_1 \).

Player 1 can raise an objection against player \( n \). Let \( Q \) be a coalition in \( \Gamma_{n1} \) such that for any objection \( (P, y) \) of 1 against \( n \),

\[
y(P \cap Q) + x(P \setminus Q) \leq f(|Q|).
\]
We denote \( q := |Q| \). Observe that due to inequality (1) we have that \( x(Q) \leq f(q) \). Since, \( x_n > x_1 \geq f(1) \) and \( n \in Q \) this in particular yields that \( q > 1 \). Furthermore, this yields that
\[
f(q) - x(\{1\} \cup Q \setminus \{n\}) > f(q) - x(Q) \geq 0.
\]

Take \( 0 < \delta < x_n - x_1 \) such that \( f(q) > x(\{1\} \cup Q \setminus \{n\}) + \delta \) and define
\[
y_i := x_i + \frac{\delta}{q-1} [f(q) - \delta - x(\{1\} \cup Q \setminus \{n\})]
\]
for all \( i \in Q \setminus \{n\} \).

Then \((P, y)\) with \( P := \{1\} \cup Q \setminus \{n\} \) is an objection from 1 against \( n \). However,
\[
y(P \cap Q) + x(Q \setminus P) = y(Q \setminus \{n\}) + x_n
\]
\[
= f(q) + x_n - x_1 - \delta > f(q).
\]

This contradicts inequality (1).

Theorem 2.8 in fact shows that on the class of symmetric TU-games the positive kernel and the reactive bargaining set coincide. The positive kernel is a solution concept, introduced by Sudhölter and Peleg (2000), which contains the kernel and the core but is, in turn, contained in the reactive bargaining set.

Next, we prove that on the class of superadditive and symmetric TU-games the semireactive bargaining set and the positive kernel also coincide. The same result also holds for superadditive simple TU-games, but for arbitrary superadditive TU-games this statement is no longer true. We refer to Sudhölter and Potters (2001) for the details.

**Theorem 2.9. Let \((N, f)\) be a superadditive and symmetric TU-game, then**
\[
\mathcal{M}_{sr}(f) = \begin{cases} 
\mathcal{C}(f) & \text{if } \mathcal{C}(f) \neq \emptyset, \\
\mathcal{K}(f) & \text{if } \mathcal{C}(f) = \emptyset.
\end{cases}
\]

**Proof.** Let \((N, f)\) be a superadditive and symmetric TU-game. Take \( x \in \mathcal{M}_{sr}(f) \). Suppose \( x \notin \mathcal{C}(f) \cup \mathcal{K}(f) \). Then according to Lemma 2.7 we have in particular \( x_n > x_1 \).

Player 1 can raise an objection against player \( n \). Take \( P := \{1, \ldots, p\} \in \Gamma_n \) such that
\[
f(p) - x(P) = \max \{ f([P']) - x(P') \mid P' \in \Gamma_n \} \text{ and } 2 \leq p \leq n - 1 \text{ is maximal}.
\]

Let \( Q \) be any coalition in \( \Gamma_n \) and denote \( q := |Q| \). We may assume that \( x(Q) \leq f(q) \), otherwise any objection of player 1 is justified.

**Claim 1.** \( f(q) - x(Q) < f(p) - x(P) \).

**Proof.** Suppose \( f(q) - x(Q) \geq f(p) - x(P) \). Define \( P' := \{1\} \cup Q \setminus \{n\} \). Since \( x_n > x_1 \), this yields
\[
f([P']) - (x_1 + \ldots + x_{|P'|}) = f(q) - (x_1 + \ldots + x_q)
\]
\[
> f(q) - x(Q) \geq f(p) - x(P).
\]

Contradiction with the definition of \( P \).
2.5 Symmetric TU-games

Claim 2. \( P \cap Q \neq \emptyset \).

Proof. Suppose that \( P \cap Q = \emptyset \). Then

\[
 f(p + q) - (x_1 + \ldots + x_{p+q}) \geq f(p + q) - x(P \cup Q) \\
\geq f(p) - x(P) + f(q) - x(Q) \geq f(p) - x(P).
\]

The second inequality follows from the superadditivity of \( (N, f) \). Hence, by the definition of \( P \), this yields \( f(p + q) - (x_1 + \ldots + x_{p+q}) = f(p) - x(P) \). But this contradicts the maximality of \( P \).

Using Claim 1 and Claim 2 we can now construct a justified objection via coalition \( P \). Take \( \delta > 0 \) such that \( 0 \leq f(q) - x(Q) < f(p) - x(P) - \delta \) and define

\[
y_i := x_i + \frac{\delta}{|P \cap Q|} \quad \text{for all } i \in P \backslash Q, \\
y_i := x_i + \frac{1}{|P \cap Q|} [f(p) - x(P) - \delta] \quad \text{for all } i \in P \cap Q.
\]

Then \( (P, y) \) is an objection from 1 against \( n \). Furthermore,

\[
y(P \cap Q) + x(Q \setminus P) = x(P \cap Q) + f(p) - x(P) - \delta + x(Q \setminus P) \\
= x(Q) - x(P) + f(P) - \delta > f(q).
\]

This means that the objection \( (P, y) \) is justified and the desired contradiction is obtained. \( \square \)

2.5.2 The bargaining set

In this subsection the bargaining set is studied on the class of symmetric TU-games. Let us start by proving the following useful lemma which states that if there is a justified objection, then there is a justified objection from player 1 against player \( n \).

Lemma 2.10. Let \( (N, f) \) be a symmetric TU-game and let \( x \in \mathcal{I}(f) \) be an imputation with \( x_1 \leq \ldots \leq x_n \). There is a justified objection with respect to \( x \) if and only if there is a justified objection \( (P, y) \) of player 1 against player \( n \) with \( P := \{1, \ldots, p\} \) and \( y_1 \leq \ldots \leq y_p \).

Proof. Let \( x \in \mathcal{I}(f) \) with \( x_1 \leq \ldots \leq x_n \). Assume that player \( k \) has a justified objection against player \( \ell \) with respect to \( x \).

Claim 1. \( k < \ell \).

Proof. Let \( (P, y) \) be a justified objection of player \( k \) against player \( \ell \). Let \( Q := \{\ell\} \cup P \setminus \{k\} \). Then

\[
y(P \cap Q) + x(Q \setminus P) = y(P) - y_k + x_\ell < f(|P|) - x_k + x_\ell.
\]

But on the other hand, since \( (P, y) \) is a justified objection, we also have that

\[
y(P \cap Q) + x(Q \setminus P) > f(|Q|) = f(|P|).
\]

Hence, \( x_k < x_\ell \). By the assumption on \( x \in \mathcal{I}(f) \), this yields that \( k < \ell \).
Claim 2. Player $k$ has a justified objection against player $n$.

Proof. Let $(P, y)$ be a justified objection of player $k$ against player $\ell$. According to Claim 1 we know that $k \neq n$. Define $\tilde{P} \in \Gamma_{kn}$ by

$$\tilde{P} := \{\ell\} \cup P \setminus \{n\} \text{ if } n \in P \text{ and } \tilde{P} := P \text{ if } n \notin P$$

and define $\tilde{y} \in \mathbb{R}_+^P$ by

$$\tilde{y}_j := y_j \text{ if } j \in \tilde{P} \setminus \{\ell\} \text{ and } \tilde{y}_\ell := y_n.$$

Observe that $\tilde{y} = y$ whenever $n \notin P$. Furthermore, we have that $\tilde{y}_\ell = y_n > x_n \geq x_\ell$ and of course that $\tilde{y}_j = y_j > x_j$ for all $j \in \tilde{P} \setminus \{\ell\}$. Hence, $(\tilde{P}, \tilde{y})$ is an objection of player $k$ against player $n$ with respect to $x \in \mathcal{I}(f)$. Next, we prove that it is a justified objection.

Suppose player $n$ is able to counter this objection by coalition $Q \in \Gamma_{nk}$. This means that

$$\tilde{y}(\tilde{P} \cap Q) + x(Q \setminus \tilde{P}) \leq f(|Q|).$$

Define $Q \in \Gamma_{\ell k}$ by

$$Q := \tilde{Q} \text{ if } \ell \in \tilde{Q} \text{ and } Q := \{\ell\} \cup \tilde{Q} \setminus \{n\} \text{ if } \ell \notin \tilde{Q}.$$

Then it can be verified that

$$y(P \cap Q) = \tilde{y}(\tilde{P} \cap Q) \text{ and } x(Q \setminus P) \leq x(Q \setminus \tilde{P}).$$

Since $f(|Q|) = f(|\tilde{Q}|)$, this yields that

$$y(P \cap Q) + x(Q \setminus P) \leq \tilde{y}(\tilde{P} \cap Q) + x(Q \setminus \tilde{P}) \leq f(|\tilde{Q}|) = f(|Q|).$$

This means that player $\ell$ is able to counter the objection $(P, y)$ of player $k$ via coalition $Q \in \Gamma_{\ell k}$. Contradiction.

So far, we can conclude that player $k$ has a justified objection against player $n$. Using a similar reasoning as in the proof of Claim 2, one can prove that also player 1 has a justified objection against player $n$. Say, player 1 has a justified objection against player $n$ using coalition $P := \{i(1), \ldots, i(p)\}$ with $1 = i(1) < \ldots < i(p)$. Define $\hat{P} := \{1, \ldots, p\}$ and define

$$\hat{y}_j := y_{i(j)} \text{ for all } 1 \leq j \leq p.$$

Then $\hat{y}_j = y_{i(j)} > x_{i(j)} \geq x_j$ for all $1 \leq j \leq p$. Furthermore, $\hat{y}(\hat{P}) = y(P) = f(p)$. So, $(\hat{P}, \hat{y})$ is also objection from player 1 against player $n$ with respect to $x$. It is straightforward to prove that for any coalition $Q \in \Gamma_{1n}$ it holds that

$$\hat{y}(\hat{P} \cap Q) = y(P \cap Q) \text{ and } x(Q \setminus \hat{P}) \geq x(Q \setminus P).$$

Hence, since player $n$ is not able to counter the objection $(P, y)$, he also cannot counter the objection $(\hat{P}, \hat{y})$.

Finally, by ordering the coordinates of $\hat{y} \in \mathbb{R}^P$ in a weakly increasing order we still obtain a justified objection from player 1 against player $n$. So, we may assume that $\hat{y}_1 \leq \ldots \leq \hat{y}_p$. This completes the proof. \qed
So, for symmetric TU-games we may assume, without loss of generality, that every (justified) objection in terms of the bargaining set is from player 1 against player n. Therefore, it is sufficient to give the (justified) objection \((P, y)\) only in which we furthermore may assume that \(P = \{1, \ldots, p\}\) and that \(y_1 \leq \ldots \leq y_p\).

Next, we give a necessary and sufficient condition for an element outside the core to be an element of the bargaining set. To do so, we define two types of values, the counter-value and the objection-value. With the help of these values one can check whether an objection is justified or not. Let us make this more formal.

Let \((N, f)\) be a balanced and symmetric TU-game. Take \(x \in \mathcal{I}(f) \setminus \mathcal{C}(f)\). Recall that \(x_1 \leq \ldots \leq x_n\). Since, \(x \notin \mathcal{C}(f)\), there exists at least one number \(2 \leq p \leq n-1\) such that \(x_1 + \ldots + x_p < f(p)\). Furthermore, this implies that \(x_n > f(n)/n > x_1\).

Let \(h \in N; f\) be a balanced and symmetric TU-game. Take \(x \in \mathcal{I}(f) \setminus \mathcal{C}(f)\). Since, \(x \notin \mathcal{C}(f)\), there exists at least one number \(2 \leq p \leq n-1\) such that \(x_1 + \ldots + x_p < f(p)\). Furthermore, this implies that \(x_n > f(n)/n > x_1\).

We assume that there exists at least one number \(2 \leq q \leq n-1\) such that \(x_n + \sum_{i=2}^{q} x_i \leq f(q)\). Otherwise, any objection of player 1 is justified. Then we define the counter-value of \(x\) by

\[
\zeta(x) := \min\{\zeta \geq x_2 \mid x_n + \sum_{i=2}^{q} \max\{\zeta, x_i\} \geq f(q)\ \text{for all}\ \forall 2 \leq q \leq n-1\}.
\]

**Lemma 2.11.** We have \(\zeta(x) < \frac{f(n)}{n}\).

**Proof.** Suppose \(\zeta(x) \geq \frac{f(n)}{n}\). Then, because \(x_n > \frac{f(n)}{n}\) and the TU-game \((N, f)\) is balanced, it follows that for all \(2 \leq q \leq n-1\),

\[
x_n + \sum_{i=2}^{q} \max\{\zeta(x), x_i\} > q \cdot \frac{f(n)}{n} \geq f(q).
\]

Contradiction with the definition of \(\zeta(x)\).

Define the vector \(z(x) \in \mathbb{R}^N\) with \(z(x) \geq x\) by

\[
z(x)_1 := x_1 \text{ and } z(x)_i := \max\{\zeta(x), x_i\} \text{ for } 2 \leq i \leq n.
\]

Furthermore, we define \(q(x) := \min\{2 \leq q \leq n-1 \mid x_n + \sum_{i=2}^{q} z(x)_i = f(q)\}\) and we define \(s(x) := \max\{2 \leq i \leq n \mid z(x)_i = \zeta(x)\}\). Hence,

\[
z(x) = (x_1, \zeta(x), \zeta(x), \ldots, \zeta(x), x_{s(x)+1}, \ldots, x_n)
\]
Bargaining Sets in Cooperative Games

Since $\zeta(x) < \frac{f(n)}{n} < x_n$ (Lemma 2.11), we have that $s(x) + 1 \leq n$. Furthermore, observe that

$$x_n + \sum_{i=2}^{q(x)} z(x)_i = f(q(x)).$$

This means that the surface of the gray area in Figure 4(i) is equal to $f(q(x))$. So, the vector $z(x)$ can be understood as an upper-bound for all improvements upon $x$ which still admit a counter-objection via coalition $\{2, \ldots, q(x), n\}$.

For all $2 \leq p \leq n-1$ we define the objection-value $\eta(p, x)$ of $x$ as,

$$\eta(p, x) := \min \{ \eta \geq x_2 \mid x_1 + \sum_{i=2}^{p} \max \{ \eta, x_i \} \geq f(p) \}.$$

Furthermore, we define the vector $y(p, x) \in \mathbb{R}^n$ with $y(p, x) \geq x$ by

$y(p, x)_i := x_1, y(p, x)_i := \max \{ \eta(p, x)_i, x_i \}$ for $2 \leq i \leq p$ and $y(p, x)_i := x_1$ for $p+1 \leq i \leq n$.

Let $t(p, x) := \max \{ 1 \leq i \leq p \mid y(p, x)_i > x_1 \}$. Hence,

$$y(p, x) = (x_1, \eta(p, x), \eta(p, x), \ldots, \eta(p, x), x_{t(p, x)+1}, \ldots, x_n)$$

Observe that $\eta(p, x) > x_2$ if and only if $x_1 + \ldots + x_p < f(p)$. Furthermore, note that,

$$x_1 + \sum_{i=2}^{p} y(p, x)_i = f(p).$$
2.5 Symmetric TU-games

This means that the gray surface in Figure 4(ii) equals \( f(p) \). So, the vector \( y(p, x) \) restricted to coalition \( \{1, \ldots, n\} \) is a (weak) improvement upon \( x \). In Figure 4(ii) all players in \( \{1, \ldots, 8\} \) strictly improve with respect to \( x \), except players 1, 7 and 8.

Given an imputation \( x \in I(f) \cap C(f) \), one can check with the help of the counter-value and the objection-values whether \( x \) admits a justified objection or not. The idea is as follows. Using a continuity argument the vector \( y(p, x) \) restricted to coalition \( \{1, \ldots, p\} \) can be transformed into an objection of player 1 against player \( n \). If \( \zeta(x) \geq \eta(p, x) \), then player \( n \) is able to counter this objection via coalition \( \{2, \ldots, q(x), n\} \). If the vector \( \zeta(x) < \eta(p, x) \), then player \( n \) is only able to counter if there are enough players left outside coalition \( \{1, \ldots, p\} \), he can use to counter. The following theorem formalizes this idea.

**Theorem 2.12.** Let \( \langle N, f \rangle \) be a balanced and symmetric TU-game and let \( x \in I(f) \setminus C(f) \).

(i) If \( \zeta(x) \geq \eta(p, x) \), then \( x \) admits no justified objection by coalition \( \{1, \ldots, p\} \).

(ii) If \( \zeta(x) < \eta(p, x) \) and \( p \geq s(x) \), then \( x \) admits a justified objection by coalition \( \{1, \ldots, p\} \).

(iii) If \( \zeta(x) < \eta(p, x) \) and \( p < s(x) \), then \( x \) admits no justified objection by coalition \( \{1, \ldots, p\} \) if and only if there exists a pair \( (q, i) \) such that

\[
1 \leq i \leq \min\{p, q\}, \quad p + q - i \leq n - 1 \quad \text{and} \quad (i - 1) \cdot \eta(p, x) + x_{p+1} + \cdots + x_{p+q-i} + x_n \leq f(q).
\]

**Proof.** (i) Let \( \zeta(x) \geq \eta(p, x) \) and \( (P, \omega) \) be an objection. Recall from Lemma 2.10 that we may assume that \( P := \{1, \ldots, p\} \) and that \( \omega_1 \leq \cdots \leq \omega_p \). Then it can be verified by induction, starting at \( i = p \), that for all \( 2 \leq i \leq p \),

\[
\omega_2 + \cdots + \omega_i < y(p, x)_2 + \cdots + y(p, x)_i.
\]

Define \( Q := \{2, \ldots, q(x), n\} \). If \( q(x) \leq p \), then

\[
\omega(Q \cap P) + x(Q \setminus P) = \omega_2 + \cdots + \omega_{q(x)} + x_n < y(p, x)_2 + \cdots + y(p, x)_{q(x)} + x_n \leq z(x)_2 + \cdots + z(x)_{q(x)} + x_n = f(q(x)).
\]

If \( q(x) > p \), then

\[
\omega(Q \cap P) + x(Q \setminus P) = \omega_2 + \cdots + \omega_p + x_{p+1} + \cdots + x_{q(x)} + x_n < y(p, x)_2 + \cdots + y(p, x)_p + x_{p+1} + \cdots + x_{q(x)} + x_n \leq z(x)_2 + \cdots + z(x)_p + z(x)_{p+1} + \cdots + z(x)_{q(x)} + x_n = f(q(x)).
\]

So, the coalition \( \{2, \ldots, q(x), n\} \) is able to counter.

(ii) Let \( \zeta(x) < \eta(p, x) \) and \( p \geq s(x) \). Recall that

\[
y(p, x) := (x_1, \underbrace{\eta(p, x), \eta(p, x), \ldots, \eta(p, x)}_{t-1}, x_{t+1}, \ldots, x_n)
\]
From this vector \( y(p, x) \) we construct the objection \((P, \omega)\) with \( P := \{1, \ldots, p\} \) and with 
\[
\omega := (x_1 + \varepsilon, \eta(p, x) - \delta, \ldots, \eta(p, x) - \delta, x_{t+1} + \varepsilon, \ldots, x_p + \varepsilon),
\]
with the numbers \( \delta > 0 \) and \( \varepsilon > 0 \) such that \( \eta(p, x) - \delta > \max\{\zeta(x), x_i\} \) for all \( 2 \leq i \leq t \) and \((p-t+1) \cdot \varepsilon = (t-1) \cdot \delta \). It turns out that it is a justified objection, as we now demonstrate.

Let \( Q \in \Gamma_{n+1} \) with \( |Q| = q \). Then:

- For \( i \in Q \) with \( 2 \leq i \leq t \) we have, \( \omega_i > z(x)_i \),
- For \( i \in Q \) with \( t < i \leq p \) we have, \( \omega_i > x_{t+1} \geq \eta(p, x) > \zeta(x) \) and \( \omega_i > x_i \). Hence, again \( \omega_i > z(x)_i \),
- For \( i \in Q \) with \( i > p \), we have \( x_i \geq x_{p+1} \geq x_{s(x)+1} > \zeta(x) \). This means \( x_i = z(x)_i \).

Let \( i := |Q \cap P| + 1 \) and thus \( |Q \setminus P| = q - i + 1 \). If \( i \geq 2 \), then
\[
\omega(Q \cap P) + x(Q \setminus P) = \omega_2 + \ldots + \omega_i + x_{p+1} + \ldots + x_{p+i-1} + x_n
\]
\[
\geq z(x)_2 + \ldots + \omega_i + z(x)_{p+1} + \ldots + z(x)_{p+i-1} + x_n
\]
\[
\geq z(x)_2 + \ldots + z(x)_i + z(x)_{p+1} + \ldots + z(x)_{p+i-1} + x_n
\]
\[
\geq f(q).
\]

And if \( i = 1 \), then
\[
\omega(Q \cap P) + x(Q \setminus P) = x(Q \setminus P) = x_{p+1} + \ldots + x_{p+i-1} + x_n
\]
\[
\geq z(x)_2 + \ldots + z(x)_q + x_n \geq f(q).
\]

The last inequality follows from the fact that \( x_{p+i} \geq x_{s(x)+i} > \zeta(x) \) and \( x_{p+i} \geq x_{n+i} \).

Hence, from both cases it follows that the objection \((P, \omega)\) is justified.

(iii) Let \( \zeta(x) < \eta(p, x) \) and \( p < s(x) \) and let \((q, i)\) such that condition \((C)\) is satisfied. Take any objection \((P, \omega)\). Then
\[
\omega_2 + \ldots + \omega_i < y(p, x)_2 + \ldots + y(p, x)_i = (i-1) \cdot \eta(p, x).
\]

Hence,
\[
\omega_2 + \ldots + \omega_i + x_{p+1} + \ldots + x_{p+i-1} + x_n < f(q).
\]

And thus the coalition \( \{2, \ldots, i, p+1, \ldots, p+q-i, n\} \) is able to counter the objection \((P, \omega)\).

Conversely, take \( \varepsilon > 0 \) such that \( \eta(p, x) - \varepsilon > \zeta(x) \) and \( \eta(p, x) - \varepsilon > x_p \). Define the objection \((P, \omega)\) as follows,
\[
\omega_1 := x_1 + (p - 1) \cdot \varepsilon,
\]
\[
\omega_i := \eta(p, x) - \varepsilon \quad \text{for all } 2 \leq i \leq p.
\]
Because the objection \((P, \omega)\) is not a justified objection it follows that there exists a coalition \(Q \in \Gamma_n\) with \(|Q \cap P| = i - 1\) such that
\[
(i - 1) \cdot (\eta(p, x) - \varepsilon) + x_{p+1} + \ldots + x_{p+q-i} + x_n \leq f(q).
\]
This is true for all sufficiently small numbers \(\varepsilon\), and therefore condition (C) holds. \(\square\)

Although Theorem 2.12 is rather technical, it can be very useful. In Section 2.6 an example is studied in which Theorem 2.12 will be very useful. In the proof of the following proposition this theorem plays also an important role.

**Proposition 2.13.** Let \((N, f)\) be a balanced TU-game and let \(x \in \mathcal{I}(f) \setminus \mathcal{C}(f)\). If \(x \in \mathcal{M}(f)\) then
\[
2 \leq q(x) \leq s(x) - 1 \quad \text{and} \quad \frac{f(s(x))}{s(x)} < \frac{f(q(x))}{q(x)}.
\]

**Proof.** Let \(x \in \mathcal{M}(f) \setminus \mathcal{C}(f)\). Note that in this case there exists a number \(2 \leq q \leq n - 1\) such that \(x_n + \sum_{i=2}^{q} x_i \leq f(q)\) and thus \(q(x)\) is well-defined. Suppose \(q(x) \geq s(x)\). Then
\[
x_1 + (s(x) - 1) \cdot \zeta(x) + x_{s(x)+1} + \ldots + x_{q(x)} < x_n + (s(x) - 1) \cdot \zeta(x) + x_{s(x)+1} + \ldots + x_{q(x)} = f(q(x)).
\]
This strict inequality implies that \(\zeta(x) < \eta(q(x), x)\). According to Theorem 2.12(ii), this means that \(x\) admits a justified objection by coalition \(\{1, \ldots, q(x)\}\). Contradiction.

So, \(2 \leq q(x) \leq s(x) - 1\). Therefore, \(x_n + (q(x) - 1) \cdot \zeta(x) = f(q(x))\). Since, \(x_n > f(n)/n\) (otherwise \(x \in \mathcal{C}(f)\)) and because \(f(n)/n \geq f(q(x))/q(x)\) (balancedness) it can be verified that
\[
\zeta(x) = \frac{f(q(x)) - x_n}{q(x) - 1} < \frac{f(q(x))}{q(x)}.
\]
On the other hand, Theorem 2.12(ii) tells us that \(\zeta(x) \geq \eta(s(x), x)\). Therefore,
\[
x_1 + (s(x) - 1) \cdot \zeta(x) \geq f(s(x)).
\]
Because \(\zeta(x) \geq x_2 \geq x_1\), the inequality above yields that \(s(x) \cdot \zeta(x) \geq f(s(x))\). Hence, we can conclude that
\[
\frac{f(s(x))}{s(x)} \leq \zeta(x) < \frac{f(q(x))}{q(x)}. \quad \square
\]

Total balancedness of a symmetric TU-game \((N, f)\) implies for the map \(s \mapsto f(s)/s\) to be weakly monotonic. Therefore, the following result can be directly derived from Proposition 2.13.

**Corollary 2.14.** If \((N, f)\) is a totally balanced and symmetric TU-game, then \(\mathcal{M}(f) = \mathcal{C}(f)\) \(\square\)

For arbitrary totally balanced TU-games the bargaining set and the core do not necessarily coincide (see e.g., the five-person market game given in Maschler (1976)).
2.5.3 Summary

In the previous two subsections we have shown that for a symmetric TU-game \( \langle N, f \rangle \) with a non-empty imputation set the following implications hold:

(i) \( \langle N, f \rangle \) is balanced \( \Rightarrow M_r(f) = C(f) \), \hspace{1cm} (Theorem 2.8)
(ii) \( \langle N, f \rangle \) is superadditive and balanced \( \Rightarrow M_{sr}(f) = C(f) \), \hspace{1cm} (Theorem 2.9)
(iii) \( \langle N, f \rangle \) is totally balanced \( \Rightarrow M(f) = C(f) \). \hspace{1cm} (Corollary 2.14)

Clearly, balancedness is a necessary assumption in (i) and (ii). In the following section we give two examples which illustrate that the assumption of superadditivity in (ii) and the assumption of total balancedness in (iii) cannot be omitted.

2.6 Examples

In this section we study two symmetric TU-games. Recall from Corollary 2.6 that the property ‘the bargaining set coincides with the core’ is a weak prosperity property. The first example illustrates that none of the (weak) prosperity properties, extendability, largeness and stability of the core implies that the (semireactive) bargaining set coincides with the core. The example also shows that the notion of superadditivity in Theorem 2.9 cannot be omitted. The second TU-game shows that the notion of total balancedness cannot be omitted in Corollary 2.14. Furthermore, this TU-game illustrates a rather surprisingly phenomenon of the bargaining set. It turns out that the bargaining set and the core coincide for all possible values of the grand coalition, except for exactly one value. There, the bargaining set is the union of the core with a finite number of imputations outside the core. So, from this example we can conclude that the property ‘the bargaining set coincides with the core’ is not a prosperity property.

Since, the property ‘the bargaining set and the core coincide’ is a weak prosperity property, it might be of interest to illustrate that the prosperity property largeness of the core does not imply for the bargaining set and the core to coincide nor for the semireactive bargaining set and the core to coincide. Because on the class of symmetric games the properties largeness, exactness and stability of the core are equivalent (Biswas, Ravindran and Parthasarathy (2000)), and therefore according to Theorem 2.2 also extendability is equivalent with largeness of the core, neither of these four properties imply that the semireactive bargaining set nor the bargaining set coincides with the core.

Example. Let \( \delta \geq 0, |N| = 20 \) and \( f_5 : \{0, 1, \ldots, 20\} \rightarrow \mathbb{R} \) be defined as follows:

| \( s \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( f_5 \) | 0 | 0 | 2 | 4 | 6 | 8 | 10 | 10.5 | 12 | 13.5 | 15 | 16.5 | 18.9 |

| \( s \) | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|
| \( f_5 \) | 21.3 | 23.7 | 26.1 | 28.5 | 30.9 | 33.3 | 35.7 | 38 + \delta |

Define \( x \in \mathcal{I}(f_1) \setminus C(f_1) \) by

\[
x_1 = \ldots = x_{11} := 1.5 \quad \text{and} \quad x_{12} = \ldots = x_{20} := 2.5.
\]
2.6 Examples

Claim. \( x \in \mathcal{M}_{sr}(f_1) \).

Proof. Observe that only player \( k \) with \( 1 \leq k \leq 11 \) can raise an objection with respect to \( x \) and that the only possible objections are by a coalition \( P \subseteq \{1, \ldots, 11\} \) such that \( |P| = 5 \) or such that \( |P| = 6 \) (only for these coalitions the core condition is violated). Assume player \( k \) makes an objection against an other player \( \ell \) via \( P \). If \( 12 \leq \ell \leq 20 \), then take \( Q' \subseteq \{1, \ldots, 11\} \setminus P \) such that \( |Q'| = 5 \) and define \( Q := Q' \cup \{\ell\} \). Let \((P, y)\) be any objection, then

\[
y(P \cap Q) + x(Q \setminus P) = x_\ell + x(Q') = 2.5 + 5 \cdot 1.5 = 10 = f_6(6).
\]

If \( 1 \leq \ell \leq 11 \), then take \( Q' \subseteq \{1, \ldots, 11\} \setminus (P \cup \{\ell\}) \) such that \( |Q'| = 4 \) and define \( Q := Q' \cup \{\ell, 20\} \). Let \((P, y)\) be any objection, then again

\[
y(P \cap Q) + x(Q \setminus P) = x_\ell + x(Q') + x_{20} = 5 \cdot 1.5 + 2.5 = 10 = f_6(6).
\]

Hence, given the coalition \( P \), player \( \ell \) can counter every objection \((P, y)\) of player \( k \) and therefore \( x \in \mathcal{M}_{sr}(f_1) \subseteq \mathcal{M}(f_1) \).

Biswa et al. (2000) contains an algorithm to compute the value \( \alpha_{LC}(f) \) of a symmetric TU-game \( \langle N, f \rangle \). Applying their algorithm yields \( \alpha_{LC}(f^0) = 38.1 \) (i.e., \( \delta = 0.1 \)). Hence, because largeness of the core is a prosperity property, the core is large if \( \delta = 1 \). However, in the symmetric TU-game \( \langle N, f_1 \rangle \) neither the semireactive bargaining set nor the bargaining set coincides with the core.

Remark.

(i) The symmetric TU-games \( \langle N, f_5 \rangle \), given in the preceding example, are not superadditive. Indeed, for every \( \delta \geq 0 \) we have that \( f_5(4) + f_5(5) > f_5(9) \). Hence, because \( \mathcal{M}_{sr}(f_1) \neq \mathcal{C}(f_1) \), we can conclude that the assumption of superadditivity in Theorem 2.9 cannot simply be omitted in order for the semireactive bargaining set and the core to coincide.

(ii) The example above also illustrates that the value \( \alpha_{MC}(v^0) \) (see Subsection 2.4.2) can be larger than the value \( \alpha_{LC}(v^0) \) (and the value \( \alpha_{sr}(v^0) \)) for certain incomplete TU-games \( \langle N, v^0 \rangle \).

The next example illustrates that the property ‘the bargaining set coincides with the core’ is neither monotone nor closed and therefore it is not a prosperity property.

Example. Let \( \delta \geq 0 \), \( |N| = 13 \) and \( f_3 : \{0, 1, \ldots, 13\} \rightarrow \mathbb{R} \) be defined as follows:

\[
\begin{array}{cccccccccccccc}
s & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
f_3 & 0 & 0 & 0 & 0 & 0 & 22 & 22 & 22 & 22 & 22 & 22 & 47 & 47 & 47 & 61.1 + \delta
\end{array}
\]

Claim. Let \( \delta \geq 0 \) and let \( x \in \mathcal{I}(f_3) \setminus \mathcal{C}(f_3) \) such that \( x_1 \leq \ldots \leq x_{13} \). Then

\( x \in \mathcal{M}(f_3) \) if and only if \( x_1 = x_7 = 2 \) and \( x_8 = x_{13} = 8 \).

Before we give a proof for this claim, observe that it implies for the bargaining set \( \mathcal{M}(f_3) \) and the core \( \mathcal{C}(f_3) \) to coincide if and only if \( \delta \neq 0.9 \). But the bargaining set \( \mathcal{M}(f_{0.9}) \)
is the union of the core $C(f_{0.9})$ with the set of all possible permutations of the imputation $(2, 2, 2, 2, 2, 2, 2, 8, 8, 8, 8, 8, 8, 8, 8, 8)$ (see Figure 5 for an impression of this phenomenon). This illustrates that the property ‘the bargaining set coincides with the core’ is neither monotone nor closed and therefore it is not a prosperity property.

Proof. \( \Rightarrow \) Let \( x \in \mathcal{I}(f_{0.9}) \) be defined by \( x_1 = x_7 := 2 \) and \( x_8 = x_{13} := 8 \). Then \( x \notin C(f_{0.9}) \). Furthermore, some straightforward calculations yield that \( \zeta(x) = \frac{1}{4}(22 - 8) = 3/2 \) and thus \( s(x) = 7 \). Using Theorem 2.12 we now prove that \( x \in \mathcal{M}(f_{0.9}) \).

If \( p \geq 7 \), then \( \eta(p, x) \leq 3 \frac{1}{2} = \zeta(x) \) (only for \( p = 10 \) we have an equality). According to Theorem 2.12(i) this means that for any \( p \geq 7 \) there is no justified objection by coalition \( \{1, \ldots, p\} \).

Furthermore, it can be verified that \( \eta(5, x) = 5 \) and that \( \eta(6, x) = 4 \). However,

\[
3 \cdot \eta(6, x) + x_7 + x_{13} = 22 = f(5) \quad \text{and} \quad 2 \cdot \eta(5, x) + x_6 + x_7 + x_{13} = 22 = f(5).
\]

So, condition (C) in Theorem 2.12(iii) is in both cases satisfied. This means that \( x \) admits also no justified objection by coalition \( \{1, \ldots, 5\} \) or by coalition \( \{1, \ldots, 6\} \). Clearly, there are no objections via \( \{1, \ldots, p\} \) for \( p \leq 4 \). So, we can conclude that \( x \in \mathcal{M}(f_{0.9}) \).

\( \Rightarrow \) The proof of the converse is long and tedious. Nevertheless, it illustrates the power of Theorem 2.12.
2.6 Examples

Let \( \delta \geq 0 \) and assume that \( x \in \mathcal{M}(f_\delta) \setminus \mathcal{C}(f_\delta) \) such that \( x_1 \leq \ldots \leq x_{13} \). Then in particular \( x_1 \leq 4.7 + \frac{\delta}{12} \leq x_{13} \).

**Claim 1.** \( q(x) = 5 \) or \( q(x) = 10 \).

**Proof.** By Proposition 2.13 and the values \( f_\delta(s) \) it immediately follows that \( q(x) \geq 5 \). Suppose \( 6 \leq q(x) \leq 9 \). Proposition 2.13 also tells us that \( s(x) > q(x) \). Hence, we obtain in this case that

\[
x_{13} + 4 \cdot \zeta(x) \geq 22 \quad \text{and} \quad x_{13} + (q(x) - 1) \cdot \zeta(x) = f_\delta(q(x)) = 22.
\]

Contradiction. On the other hand, if \( 11 \leq q(x) \leq 47 \), then we obtain (again using the fact that \( s(x) > q(x) \))

\[
x_{13} + 9 \cdot \zeta(x) \geq 47 \quad \text{and} \quad x_{13} + (q(x) - 1) \cdot \zeta(x) = f_\delta(q(x)) = 47.
\]

Again a contradiction. So, \( q(x) = 5 \) or \( q(x) = 10 \). This proves Claim 1.

Since \( s(x) > q(x) \) (Proposition 2.13), we can conclude from Claim 1 and the definitions of \( \zeta(x) \) and \( q(x) \) that

\[
\zeta(x) = \begin{cases} 
\frac{1}{4} 
(22 - x_{13}) & \text{if } q(x) = 5 \\
\frac{1}{5} 
(47 - x_{13}) & \text{if } q(x) = 10.
\end{cases}
\]

As \( x_1 < x_{13} \), it follows that \( \eta(5, x) > \zeta(x) \) if \( q(x) = 5 \) and that \( \eta(10, x) > \zeta(x) \) if \( q(x) = 10 \). So, by definition of \( \eta(5, x) \) and \( \eta(10, x) \) this implies that

\[
\eta(5, x) = \frac{1}{4} 
(22 - x_1) & \text{if } q(x) = 5 \\
\eta(10, x) = \frac{1}{5} 
(47 - x_1) & \text{if } q(x) = 10.
\]

Given these statements on \( q(x), \zeta(x), \eta(5, x) \) and \( \eta(10, x) \), we now investigate which values \( s(x) \) can attain.

**Claim 2.** \( s(x) = 7 \).

**Proof.** Let \( x \in \mathcal{M}(f_\delta) \setminus \mathcal{C}(f_\delta) \). Since \( q(x) = 5 \) or \( q(x) = 10 \) (Claim 1), we immediately obtain by Proposition 2.13 that \( s(x) \notin \{2, 3, 4, 5\} \). Furthermore, \( f_\delta(10)/10 \geq f_\delta(s)/s \) for all \( 1 \leq s \leq 12 \). So, again by Proposition 2.13, it follows that \( s(x) \neq 10 \). For all the remaining possibilities, except \( s(x) = 7 \), we prove that \( x \) admits a justified objection, either by coalition \( \{1, \ldots, 5\} \) or by coalition \( \{1, \ldots, 10\} \).

- Suppose \( s(x) \in \{11, 12\} \). According to Claim 1, we have \( q(x) = 5 \) or \( q(x) = 10 \). However, if \( q(x) = 5 \) then \( \zeta(x) = \frac{1}{4} 
(22 - x_{13}) \). By definition of \( \zeta(x) \), we know that \( x_{13} + 9 \cdot \zeta(x) \geq f_\delta(10) = 47 \). This implies that \( x_{13} \leq 5 \). So, \( q(x) \neq 5 \).

Hence, if \( s(x) \in \{11, 12\} \), then \( q(x) = 10 \) which, in turn, means that \( \zeta(x) = \frac{1}{5} 
(47 - x_{13}) \) and that \( \eta(10, x) = \frac{1}{5} 
(47 - x_1) \). We demonstrate that in this case \( x \) admits a justified objection by coalition \( \{1, \ldots, 10\} \).

Observe that \( \zeta(x) < \eta(10, x) \) and \( q(x) = 10 < s(x) \). So, according to Theorem 2.12(iii), \( x \) admits no justified objection by coalition \( \{1, \ldots, 10\} \) if and only if there exists a pair \( (q, i) \) such that condition \( (C) \) is satisfied. Due to the values of \( f_\delta(s) \) for all \( 2 \leq s \leq 12 \) there are two main candidates for \( q \), namely, \( q = 5 \) or \( q = 10 \) (i.e., player 13 counts via a coalition of size 5 or via a coalition of size 10). Furthermore, the best candidate for \( i \) has to be such
that player 13 can take a player in \{11, 12\} whenever \(x_{11} < \eta(10, x)\) or \(x_{12} < \eta(10, x)\). Differently, according to Theorem 2.12(iii), player 13 is able to counter if

\[
2 \cdot \eta(10, x) + \eta(10, x) \land x_{11} + \eta(10, x) \land x_{12} + x_{13} \leq f_5(5) = 22, \quad (C_5)
\]

or if

\[
7 \cdot \eta(10, x) + \eta(10, x) \land x_{11} + \eta(10, x) \land x_{12} + x_{13} \leq f_5(10) = 47. \quad (C_{10})
\]

We denote \(a \land b := \min\{a, b\}\). If \(s(x) = 12\), then, by definition, we have that \(\eta(10, x) > \zeta(x) \geq x_{12} \geq x_{11}\). Recall that \(\eta(10, x) > 4.7\) (since \(x_1 < 4.7\)) and observe that \(x_{11} + x_{12} + x_{13} = x(N) - (x_1 + \ldots + x_{10}) > 61.1 - 47 = 14.1\). Combining these observations, we can conclude that neither equation \((C_5)\) nor equation \((C_{10})\) holds whenever \(s(x) = 12\). Differently, \(x\) admits a justified objection by coalition \(\{1, \ldots, 10\}\) whenever \(s(x) = 12\).

Similarly, if \(s(x) = 11\), then \(\eta(10, x) > \zeta(x) \geq x_{11}\). Now we assume that \(x_{12} > \eta(10, x)\) (otherwise we refer to the case \(s(x) = 12\)). Then equation \((C_5)\) implies (again we use the fact that \(\eta(10, x) > 4.7\))

\[
x_{11} + x_{13} \leq 22 - 3 \cdot \eta(10, x) < 7.9
\]

and equation \((C_{10})\) implies

\[
x_{11} + x_{13} \leq 47 - 8 \cdot \eta(10, x) < 9.4.
\]

In both cases we obtain a contradiction since,

\[
x(N) \leq 6 \cdot (x_{11} + x_{13}) + x_1 < 6 \cdot 9.4 + 4.7 + \frac{a}{13} \leq f_5(N).
\]

This means again that neither equation \((C_5)\) nor equation \((C_{10})\) is satisfied. Differently, if \(s(x) = 11\), then \(x\) also admits a justified objection by coalition \(\{1, \ldots, 10\}\).

- Suppose \(s(x) \in \{8, 9\}\). According to Proposition 2.13 and Claim 1 we know that \(q(x) = 5\) and thus \(\zeta(x) = \frac{1}{2} \cdot (22 - x_{13})\). If \(s(x) = 9\), we can conclude that

\[
x_1 + 8 \cdot \zeta(x) + x_{10} \leq x_{13} + 8 \cdot \zeta(x) + x_{13} = 44 < f_5(10).
\]

And if \(s(x) = 8\) we obtain the strict inequality

\[
x_1 + 7 \cdot \zeta(x) + x_9 + x_{10} \leq 8 \cdot \zeta(x) + 2 \cdot x_{13} = 44 < f_5(10).
\]

Hence, in both cases we obtain from the definition of \(\eta(10, x)\) that \(\eta(10, x) > \zeta(x)\). However, in both cases \(10 > s(x)\). Combining these two observations with Theorem 2.12(ii), we can conclude that \(x\) admits a justified objection by coalition \(\{1, \ldots, 10\}\) whenever \(s(x) \in \{8, 9\}\).

- Suppose \(s(x) = 6\). Then \(q(x) = 5\) and thus \(\zeta(x) = \frac{1}{4} \cdot (22 - x_{13})\) and \(\eta(5, x) = \frac{1}{4} \cdot (22 - x_1)\). We demonstrate that in this case \(x\) admits a justified objection by coalition \(\{1, \ldots, 5\}\).

Observe that \(\zeta(x) < \eta(5, x)\) and \(q(x) = 5 < s(x)\). So, according to Theorem 2.12(iii), \(x\) admits no justified objection by coalition \(\{1, \ldots, 5\}\) if and only if there exists a pair \((q, i)\)
such that condition (C) is satisfied. Similarly, as in the case of $s(x) \in \{11, 12\}$, there are two candidates for $q$, namely, $q = 5$ or $q = 10$. Furthermore, $i$ can be taken such that player 13

takes those players outside coalition $\{1, \ldots, 5\}$ whose payoff according to $x$ is less or equal than $\eta(5, x)$. This means that, according to Theorem 2.12(iii), player 13 is able to counter if

\[ x_6 + \eta(5, x) \land x_7 + \eta(5, x) \land x_8 + \eta(5, x) \land x_9 + x_{13} \leq 22, \tag{C_5} \]

(since $x_6 \leq \zeta(x) < \eta(5, x)$) or if

\[ 2 \cdot \eta(5, x) + x_6 + \ldots + x_{10} + \eta(5, x) \land x_{11} + \eta(5, x) \land x_{12} + x_{13} \leq 47 \tag{C_{10}} \]

(since there are exactly four coordinates in $g(5, x)$ equal to $\eta(5, x)$).

First, we prove that if equation (C_{10}) holds, then $x$ admits a justified objection by coalition $\{1, \ldots, 10\}$. Because $\eta(5, x) > \zeta(x)$, $x_{13} \geq x_{12} \geq x_{11} > \zeta(x)$ and $x_6 \geq x_1$, equation (C_{10}) in particular yields that

\[ x_1 + 5 \cdot \zeta(x) + x_7 + x_8 + x_9 + x_{10} < 47 = f_\delta(10). \]

But this means, by definition of $\eta(10, x)$, that $\eta(10, x) > \zeta(x)$. Since $10 > s(x) = 6$, Theorem 2.12(ii) tells us that $x$ admits a justified objection by coalition $\{1, \ldots, 10\}$. So, equation (C_{10}) does not hold.

Next, we prove that equation (C_5) does not hold either. To do so, we distinguish between the following two cases:

Case (1). Assume that $x_7 \geq \eta(5, x)$. Then equation (C_5) becomes

\[ x_6 + 3 \cdot \eta(5, x) + x_{13} \leq 22. \]

But, since $x_6 \geq x_1$ and $x_{13} \geq x_7 \geq \eta(5, x)$, this inequality implies that

\[ 22 = x_1 + 4 \cdot \eta(5, x) \leq x_6 + 3 \cdot \eta(5, x) + x_{13} \leq 22. \]

Hence, $x_1 = \ldots = x_6$ and $x_7 = \ldots = x_{13} = \eta(5, x)$. But then

\[ x_1 + 5 \cdot \zeta(x) + x_7 + \ldots + x_{10} \leq x_1 + 4 \cdot \zeta(x) + 4 \cdot \eta(5, x) + x_{13} = 22 + 22 = 44 < f_\delta(10). \]

This means again that $\eta(10, x) > \zeta(x)$. So, if equation (C_5) holds and $x_7 \geq \eta(5, x)$, then $x$ admits a justified objection by coalition $\{1, \ldots, 10\}$ (Theorem 2.12(ii)).

Case (2). Assume that $x_7 < \eta(5, x)$. Now, we already assume that $\zeta(x) \geq \eta(10, x)$ (otherwise $x$ admits a justified objection by $\{1, \ldots, 10\}$ (Theorem 2.12(ii)) and we are done).

Recall, that by definition of $\zeta(x)$, this means that

\[ x_1 + 5 \cdot \zeta(x) + x_7 + x_8 + x_9 + x_{10} \geq f_\delta(10) = 47. \]

Because $4 \cdot \zeta(x) + x_{13} = 22$, $x_6 \geq x_1$ and $x_{13} \geq x_{10} \geq x_9$, it can be verified from the equation above that

\[ x_6 + x_7 + x_8 + \zeta(x) + x_{13} \geq 25. \]
So, if $\zeta(x) \geq \eta(10, x)$ (and thus $x$ does not admit a justified objection by $\{1, \ldots, 10\}$), then equation (D) holds. Since we are in the case that $x_7 < \eta(5, x)$, equation (C5) becomes

$$x_6 + x_7 + \eta(5, x) \land x_8 + \eta(5, x) \land x_9 + x_{13} \leq 22.$$ 

Combining this equality with equation (D) it can be verified that

$$x_8 + \zeta(x) > \eta(5, x) \land x_8 + \eta(5, x) \land x_9 + 3.$$

Note that $\eta(5, x) \land x_9 > \zeta(x)$ (since, $\eta(5, x), x_9 > \zeta(x)$). Combining this observation with the strict inequality above, we immediately can conclude that $x_8 > \eta(5, x)$. Hence,

$$x_8 + \zeta(x) > \eta(5, x) \land x_8 + \eta(5, x) \land x_9 + 3 = 2 \cdot \eta(5, x) + 3.$$ 

However, since $x_8 > \eta(5, x)$ equation (C5) in fact equals, $x_6 + x_7 + 2 \cdot \eta(5, x) + x_{13} \leq 22$. But now we can derive a contradiction. Indeed,

$$x_6 + x_7 + 2 \cdot \eta(5, x) + x_{13} \geq x_1 + \zeta(x) + 2 \cdot \eta(5, x) + x_9 > x_1 + 2 \cdot \eta(5, x) + 2 \cdot \eta(5, x) + 3 = 22 + 3 = 25.$$ 

The strict inequality follows from $x_8 + \zeta(x) > 2 \cdot \eta(5, x) + 3$. Thus, if $x_7 < \eta(5, x)$, then $x$ either admits a justified objection by $\{1, \ldots, 5\}$ or by $\{1, \ldots, 10\}$ since equation (C5) and equation (D) cannot both be true. This completes the proof of Claim 2.

According to Claim 2 we know that $x$ admits a justified objection whenever $s(x) \neq 7$. So, there is only one more value left for $s(x)$ which we need to investigate. As it turns out, the imputation $x \in M(f_3) \setminus C(f_3)$ is completely determined if $s(x) = 7$.

**Claim 3.** If $s(x) = 7$, then $x_1 = \ldots = x_7 = 2$ and $x_8 = \ldots = x_{13} = 8$.

**Proof.** Let $s(x) = 7$. Then $q(x) = 5$ and thus $\zeta(x) = \frac{1}{7}(22 - x_{13})$ and $\eta(5, x) = \frac{1}{5}(22 - x_1)$. Because $5 < s(x)$ and $\zeta(x) < \eta(5, x)$ Theorem 2.12(iii) tells us now that $x$ does not admit a justified objection by coalition $\{1, \ldots, 5\}$ if

$$x_6 + x_7 + \eta(5, x) \land x_8 + \eta(5, x) \land x_9 + x_{13} \leq 22,$$ 

(since $\eta(5, x) > \zeta(x)$ or $x_7 \geq x_6$ or if

$$2 \cdot \eta(5, x) + x_6 + \ldots + x_{10} + \eta(5, x) \land x_{11} + \eta(5, x) \land x_{12} + x_{13} \leq 47$$ 

(since there are exactly four coordinates in $y(5, x)$ equal to $\eta(5, x)$).

Furthermore, Theorem 2.12(ii) tells us that $\zeta(x) \geq \eta(10, x)$ (otherwise there is a justified objection via coalition $\{1, \ldots, 10\}$). By definition of $\zeta(x)$ this means that

$$x_1 + 6 \cdot \zeta(x) + x_8 + x_9 + x_{10} \geq 47.$$ 

Because $4 \cdot \zeta(x) + x_{13} = 22$ and $x_{13} \geq x_{10}$, we can derive from this inequality

$$x_1 + x_8 + x_9 + 2 \cdot \zeta(x) \geq 25.$$ 

So, if $\zeta(x) \geq \eta(10, x)$ (and thus $x$ does not admit a justified objection via $\{1, \ldots, 10\}$), then equation (D) holds. With the help of this equation (D) we first prove that if equation (C10) holds then equation (C5) also holds.
Assume that equation \((C_{10})\) holds. Because \(\eta(5, x) > \zeta(x)\) and \(x_{10} \geq x_1\), equation \((D)\) yields in particular that
\[
x_8 + x_9 + x_{10} + 2 \cdot \eta(5, x) > 25.
\]

By subtracting this strict inequality from equation \((C_{10})\), we obtain that
\[
x_6 + x_7 + \eta(5, x) \land x_{11} + \eta(5, x) \land x_{12} + x_{13} \leq 47 - 25 = 22.
\]

Since, \(x_{12} \geq x_9\) and \(x_{11} \geq x_8\) it follows that equation \((C_5)\) also holds.

Next, we prove that if equation \((C_5)\) holds, then \(x_1 = x_7 = 2\) and \(x_8 = x_{13} = 8\). To do so, we distinguish between the following two cases:

Case (1). Assume that \(x_1 < 2\). Then \(x\) admits a justified objection by coalition \(\{1, \ldots, 6\}\) as we now demonstrate.

Suppose \(\eta(6, x) \leq \zeta(x)\). This means that \(x_1 + 5 \cdot \zeta(x) \geq 22\) and thus \(\zeta(x) > 4\). By definition of \(\zeta(x)\), this yields that \(x_{13} < 6\). Furthermore, \(\zeta(x) = \frac{1}{4}(22 - x_{13}) < 4\frac{13}{40}\) (since \(x_{13} > 4.7\)). By taking these two upper-bounds on \(x_{13}\) and \(\zeta(x)\) we obtain that
\[
x_1 + x_8 + x_9 + 2 \cdot \zeta(x) < 2 + 2 \cdot 6 + 2 \cdot 4\frac{13}{40} < 25.
\]

But this contradicts equation \((D)\). Differently, if \(\eta(6, x) > \zeta(x)\), then equation \((D)\) is violated and thus \(\zeta(x) < \eta(10, x)\) which means that \(x\) admits a justified objection via \(\{1, \ldots, 10\}\).

So, we may assume that \(\eta(6, x) > \zeta(x)\). This means that \(x\) admits an objection via coalition \(\{1, \ldots, 6\}\) and that \(\eta(6, x) = \frac{1}{8}(22 - x_1)\). Because, \(s(x) = 7\) and \(\eta(6, x) > \zeta(x) \geq x_7\), Theorem 2.12(iii) tells us that player 13 is only able to counter if
\[
x_7 + \eta(6, x) \land x_8 + \eta(6, x) \land x_9 + \eta(6, x) \land x_{10} + x_{13} \leq 22, \quad (C_5^1)
\]

or if
\[
3 \cdot \eta(6, x) + x_7 + \ldots + x_{10} + \eta(6, x) \land x_{11} + \eta(6, x) \land x_{12} + x_{13} \leq 47 \quad (C_5^{10})
\]

(since there are exactly five coordinates in \(\eta(6, x)\) equal to \(\eta(6, x)\)).

Recall that \(\zeta(x) \geq \eta(10, x)\) (otherwise there is a justified objection via \(\{1, \ldots, 10\}\) (Theorem 2.12(ii)). Therefore, equation \((D)\) holds. Using this equation we first prove that equation \((C_5^1)\) holds whenever equation \((C_{10})\) holds.

Because \(\eta(6, x) > \zeta(x) > x_1\), equation \((D)\) also yields that \(x_9 + x_{10} + 3 \cdot \eta(6, x) > 25\).

By subtracting this expression from equation \((C_{10}^1)\) and using the fact that \(x_{12} \geq x_{10}\) and \(x_{11} \geq x_9\) it follows that equation \((C_{10}^1)\) holds whenever equation \((C_5^1)\) holds.

However, using equation \((D)\) we now prove that equation \((C_5^1)\) is violated. Equation \((D)\) in particular yields that
\[
x_7 + x_8 + x_{13} + 2 \cdot \zeta(x) \geq 25.
\]

Because \(\eta(6, x) \land x_{10} \geq \eta(6, x) \land x_9 > \zeta(x)\) equation \((C_5^1)\) yields that
\[
x_7 + \eta(6, x) \land x_8 + 2 \cdot \zeta(x) + x_{13} < 22.
\]
Hence, \( x_8 \geq \eta(6, x) \land x_8 + (25 - 22) \) and thus \( \eta(6, x) < x_8 \). This means that equation (C^*_6) in fact states that
\[
x_{13} + \frac{3}{4} x_1 + 13 \frac{1}{2} = x_{13} + x_1 + 3 \cdot \eta(6, x) \leq x_{13} + x_7 + 3 \cdot \eta(6, x) \leq 22.
\]

Furthermore, observe that equation (D) in particular states that
\[
x_1 + 11 + 1 \frac{1}{2} x_{13} = x_1 + 2 \cdot x_{13} + 2 \cdot \zeta(x) \geq x_1 + x_8 + x_9 + 2 \cdot \zeta(x) \geq 25.
\]

From these two inequalities it is straightforward to verify that \( x_1 \geq 2 \). But this contradicts the assumption that \( x_1 < 2 \). Hence, equation (C^*_7) nor equation (C^*_8) holds. Differently, if \( s(x) = 7 \) and \( x_1 < 2 \), then \( x \) admits a justified objection by coalition \( \{1, \ldots, 6\} \).

Case (2). So, we are left with the case that \( x_1 \geq 2 \). In this case we demonstrate that equation (C_5) and equation (D) hold only if \( x_1 = x_7 = 2 \) and \( x_8 = x_{13} = 8 \).

Observe that \( \eta(5, x) \land x_9 \geq \eta(5, x) \land x_8 > \zeta(x), x_6 \geq x_1 \) and \( x_{13} \geq x_9 \). Therefore, equation (D) yields that
\[
x_6 + x_8 + x_{13} + \eta(5, x) \land x_8 + \eta(5, x) \land x_9 > 25.
\]

Recall that equation (C_5) states that \( x_6 + x_7 + \eta(5, x) \land x_8 + \eta(5, x) \land x_9 + x_{13} \leq 22 \). So, if equation (C_5) holds, then \( x_8 > x_7 + (25 - 22) = x_7 + 3 \). Since, we are in the case that \( x_1 \geq 2 \), this yields that \( x_8 > 5 \geq \eta(5, x) \). Hence, \( \eta(5, x) \land x_9 = \eta(5, x) \land x_8 = \eta(5, x) \).

But then equations (C_5) and (D) state that
\[
x_{13} + 2 \cdot x_6 + \frac{3}{4} (22 - x_8) \leq x_{13} + x_6 + x_7 + 2 \cdot \eta(5, x) \leq 22,
\]
\[
x_6 + 2 \cdot x_{13} + \frac{3}{4} (22 - x_{13}) \geq x_1 + x_8 + x_9 + 2 \cdot \zeta(x) \geq 25.
\]

Hence, \( x_{13} + \frac{3}{4} x_6 \leq 11 \) and \( x_6 + 1 \frac{1}{2} x_{13} \geq 14 \). From these two observations one can verify that \( x_6 \leq 2 \). Since \( x_1 \geq 2 \), this yields that \( x_1 = x_6 = 2 \). From this it is straightforward to verify by these two previous inequalities that \( x_1 = x_7 = 2 \) and that \( x_8 = x_{13} = 8 \).

Remark.

(i) The symmetric TU-game \( \langle N, f_3 \rangle \), given in the previous example, is for every \( \delta \geq 0 \) superadditive. Hence, the assumption of total balancedness in Corollary 2.14 cannot be weakened to superadditivity.

(ii) It is left to the reader to check that \( \alpha_{MC}(f_0^I) = 117.5 \) (i.e., \( \mathcal{I}(f_3) = \bigcup_{I \subseteq N \mathcal{A}_I(f_3)} \) whenever \( \delta > 56.4 \)). The algorithm of Biswas et al. (2000) tells us that \( \alpha_{CC}(f_0^I) = 198 \) (i.e., the core \( \mathcal{C}(f_3) \) is large whenever \( \delta \geq 136.9 \)). Hence, \( \alpha_{MC}(v^0) < \alpha_{CC}(v^0) \) for certain incomplete TU-games \( \langle N, v^0 \rangle \).

(iii) According to Biswas et al., the core \( \mathcal{C}(f_{0, 9}) \) is not stable (see Subsection 2.2). Indeed, any of the imputations outside the core of the type \( \langle 2, \ldots, 2, 8, \ldots, 8 \rangle \) cannot be dominated by a core allocation.
3

Bargaining Sets and Price Equilibria in Exchange Economies

3.1 Introduction

The main part of this chapter defines and investigates the reactive and the semireactive bargaining set in exchange economies with indivisible goods and one perfectly divisible good (money). This part of the chapter is adapted from Meertens, Potters and Reijnierse (2005). Additionally, the final section, based on Meertens (2005), illustrates that every non-negative, superadditive and balanced TU-game generates an exchange economy with a price equilibrium.

Exchange economies, as investigated in this chapter, find their origin in Debreu (1959) (see also Debreu (1983)). In these economies a group of people, called agents, can buy and sell indivisible goods (e.g., houses, cars) by payments made in units of one infinitely divisible good that, following standard use, is referred to as money. Each agent ‘arrives at the market’ with an amount of money and a number of indivisible goods. Moreover, each agent has an appreciation for the indivisible goods and the amounts of money. Since, we allow these agents to obtain more than one good, they have preferences on consumption bundles consisting of a number of objects and an amount of money. The preferences are assumed to be complete, transitive and continuous binary relations on the set of consumption bundles under the four additional assumptions that ‘more money is better than less money’, ‘large amounts of money can change preferences’, ‘indivisible goods are weakly desired’ and that ‘the marginal utility for money is constant’. These preference relations can be represented by quasi-linear utility functions.

Exchange economies with indivisible goods, money and quasi-linear utilities have been stud-
Bargaining Sets and Price Equilibria in Exchange Economies

ied in so many papers, one cannot even hope to give an overview. We therefore only mention Beviá, Quinzii and Silva (1999), Bikhchandani and Mamer (1997) and the more recent paper by Potters, Reijnierse and Gellekom (2002), since we use some of their results. The main problems studied in the literature on these type of exchange economies are the existence of price (i.e., Walrasian) equilibria and/or the existence of core reallocations. In the first part of this chapter, however, we introduce and investigate other solution concepts for these economies. Since, the reactive and the semireactive bargaining set, investigated in Chapter 2 for TU-games, are defined in ordinal terms, they can be extended to exchange economies with indivisible goods and money. This is done in the first part of the present chapter.

As in the theory of TU-games the reactive bargaining set of an exchange economy is a subset of the semireactive bargaining set and both bargaining sets contain the core whenever the latter is non-empty. Given an abundance condition on the amounts of money agents initially have (the so-called Total Abundance (TA) condition), we prove that the (semi)reactive bargaining set is non-empty, even if the core is empty. However, if the exchange economy does not satisfy the TA-condition or if the utilities are not quasi-linear, then the reactive and the semireactive bargaining set may be empty. Furthermore, it is shown that in a reallocation of the (semi)reactive bargaining set, each agent receives a bundle which he appreciates at least as much as his initial endowment. Differently, reallocations in one of these two bargaining sets are individual rational. This result does not hold for the bargaining set of Aumann and Maschler (1964).

By assigning to every exchange economy, satisfying the TA-condition, a non-negative and superadditive TU-game, we prove that every reallocation in the (semi)reactive bargaining set of the economy generates an imputation in the (semi)reactive bargaining set of the TU-game. Also this result does not hold for the bargaining set. That is, we show that reallocations in the bargaining set of the economy may not even generate efficient allocations of the corresponding TU-game. Furthermore, the (semi)reactive bargaining set and the (strong) core of an exchange economy, satisfying the TA-condition, coincide if and only if the (semi)reactive bargaining set and the core of the corresponding TU-game coincide. An interesting observation is that every non-negative and superadditive TU-game gives rise to an exchange economy with quasi-linear utility functions. Recall from Chapter 2 that there are several structured classes of superadditive TU-games for which the core coincides with the (semi)reactive bargaining set. Therefore, each of these classes of superadditive TU-games generates exchange economies in which the (semi)reactive bargaining set and the (strong) core coincide.

The observation that every non-negative and superadditive TU-game gives rise to an exchange economy is the starting point for the second part of the chapter. Here, we prove that such a TU-game generates an exchange economy with a price equilibrium whenever it is balanced. However, different economies may yield the same TU-game. So, it is possible that the set of price equilibria of two exchange economies, yielding the same TU-game, may be different. This illustrates that the concept of price equilibrium is not a game theoretical solution concept, i.e., it is a solution rule for economies, but not for the corresponding games.

Let us start by fixing the terminology used in this chapter and by deriving some basic results.
3.2 Preliminaries and some basic results

An exchange economy with indivisible goods and money, studied in this chapter and denoted by $\mathcal{E}$, has the following features:

- There is a finite set of agents $N$, with $n := |N| \geq 2$.
- There is a finite set of indivisible goods $\Omega$, with $|\Omega| \geq 1$.
- Each agent $i \in N$ has an initial endowment $(A_i, m_i)$, in which $A_i \subseteq \Omega$ denotes the collection of indivisible goods initially held by agent $i$ and $m_i > 0$ denotes the strictly positive amount of money agent $i$ has in the beginning. The list $(A_i)_{i \in N}$ is an $N$-distribution of $\Omega$, i.e., $\bigcup_{i \in N} A_i = \Omega$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. We allow, however, $A_i = \emptyset$ for some agents $i \in N$.
- Each agent $i \in N$ has a preference relation $\preceq_i$ on the set $2^\Omega \times \mathbb{R}_+$ which has the following properties:
  
  1. $\preceq_i$ is a complete, transitive binary relation on $2^\Omega \times \mathbb{R}_+$.
  2. For all bundles $(B, x)$ and $(C, y)$ with $(B, x) \succ_i (C, y)$ and $x > 0$ there exists a strictly positive number $\delta$ such that $(B, x - \delta) \succ_i (C, y)$ and $(B, x) \succ_i (C, y + \delta)$ and if $(B, 0) \succ_i (C, y)$ for certain $B, C \subseteq \Omega$ and $y \in \mathbb{R}_+$, then there exists a strictly positive number $\delta$ such that $(B, 0) \succ_i (C, y + \delta)$ (continuity in money).
  3. For all $B \subseteq \Omega$, if $x > y$, then $(B, x) \succ (B, y)$ (strict monotonicity in money).
  4. $(B, 0) \succeq (\emptyset, 0)$ for all $B \subseteq \Omega$ (weak desirability for goods).
  5. For every consumption bundle $(B, x)$ there exists a positive number $\Delta$ such that $(B, x) \preceq_i (\emptyset, \Delta)$ (archimedean property).
  6. For all $B \subseteq \Omega$ and every $x \in \mathbb{R}_+$ with $(B, 0) \sim_i (\emptyset, x)$ it holds that $(B, d) \sim_i (\emptyset, x + d)$ for every $d \in \mathbb{R}_+$ (marginal utility of money is constant).

The set of all preference relations satisfying the Properties (i)–(vi) is denoted by $\mathcal{P}$. Property (iv) states that small changes in money do not change the preference. According to Property (iii) more money is better than less money and Property (iv) states that the indivisible goods are weakly desired objects. Observe that the latter property is weaker than ‘weak monotonicity in goods’ (i.e., for all $x \in \mathbb{R}_+$ and $i \in N$ it holds that $(B, x) \succeq_i (C, x)$ whenever $C \subseteq B$), a property quite common in the literature. Property (v) states that large amounts of money can change preferences and Property (vi) states that money can be used to transfer utility from one agent to another, since the marginal utility of money does not depend on the agent nor on his wealth.

We start this section by proving that each preference relation $\preceq \in \mathcal{P}$ can be presented by a quasi-linear utility function.

Proposition 3.1. For each preference relation $\preceq \in \mathcal{P}$ there exists a unique map $V: 2^\Omega \rightarrow \mathbb{R}_+$ with the properties:

- $V(\emptyset) = 0$
- $(B, x) \preceq (C, y)$ if and only if $V(B) + x \leq V(C) + y$. 

Proof. We prove that

for every $B \subseteq \Omega$ and every $x \in \mathbb{R}_+$ there exists exactly one real number $u_B(x) \in \mathbb{R}_+$ such that $(B, x) \sim (\emptyset, u_B(x))$.

Let $B \subseteq \Omega$ and $x \in \mathbb{R}_+$. Define $\mathcal{K} := \{ y \in \mathbb{R}_+ \mid (B, x) \preceq (\emptyset, y) \}$ and $\mathcal{G} := \{ y \in \mathbb{R}_+ \mid (B, x) \succeq (\emptyset, y) \}$. Then $\mathcal{K}$ is non-empty according to the archimedean property. Also $\mathcal{G}$ is non-empty since, according to strict monotonicity in money and weak desirability for goods, we have that $0 \in \mathcal{G}$. Moreover, both sets are closed according to continuity in money. By completeness of $\preceq$ we have $\mathcal{K} \cup \mathcal{G} = \mathbb{R}_+$. Hence $\mathcal{K} \cap \mathcal{G} \neq \emptyset$.

If $y_1, y_2 \in \mathcal{K} \cap \mathcal{G}$, then $(\emptyset, y_1) \sim (B, x) \sim (\emptyset, y_2)$. According to transitivity of $\preceq$, this yields, $(\emptyset, y_1) \sim (\emptyset, y_2)$ and therefore, by strict monotonicity in money, we have $y_1 = y_2$. Hence, for every $B \subseteq \Omega$ and $x \in \mathbb{R}_+$ there exists a unique number $u_B(x) \in \mathbb{R}_+$ such that $(B, x) \sim (\emptyset, u_B(x))$.

Define $V(B) := u_B(0)$ for all $B \subseteq \Omega$, then $V : 2^\Omega \rightarrow \mathbb{R}_+$ is uniquely determined and $V(\emptyset) = 0$. Furthermore, let $B \subseteq \Omega$ and $x \in \mathbb{R}_+$. By constant marginal utility in money, we have that $(\emptyset, u_B(0) + x) \sim (B, 0 + x) \sim (\emptyset, u_B(x))$. Hence, by transitivity and strict monotonicity in money, we have that $u_B(x) = u_B(0) + x = V(B) + x$ for all $B \subseteq \Omega$ and $x \in \mathbb{R}_+$. From this observation it immediately follows that the map $V : 2^\Omega \rightarrow \mathbb{R}_+$ also satisfies property (ii) mentioned in the proposition. \hfill \Box

Remark. Conversely, if $\preceq$ is a binary relation on $2^\Omega \times \mathbb{R}_+$ that can be represented by the utility function $U(B, x) := V(B) + x$, in which $V : 2^\Omega \rightarrow \mathbb{R}_+$ with $V(\emptyset) = 0$, then $\preceq$ belongs to $\mathcal{P}$.

The value $V_i(B) \in \mathbb{R}_+$ is the reservation value of agent $i \in N$ for $B \subseteq \Omega$. Due to the assumptions on the preference relations, an exchange economy $E$ can be described by the tuple $(\mathcal{N}, \Omega, (A_i, m_i, V_i)_{i \in \mathcal{N}})$.

In an exchange economy a coalition of agents can exchange their initial endowments. To put it differently, the indivisible goods and the amounts of money can be reallocated among a coalition of agents. Therefore, we introduce the following definitions in this context.

Definition. Let $S \subseteq \mathcal{N}$. An $S$-redistribution is a list $(B_i)_{i \in S}$ with $\bigcup_{i \in S} B_i = \bigcup_{i \in S} A_i$ and $B_i \cap B_j = \emptyset$ whenever $i \neq j$. So, we allow that $B_i = \emptyset$ for some agents $i \in S$. If $(B_i)_{i \in S}$ is an $S$-redistribution and $x \in \mathbb{R}_+^S$ is such that $\sum_{i \in S} x_i = \sum_{i \in S} m_i$, then the list $(B_i, x_i)_{i \in S}$ is an $S$-reallocation.

Remark. The assumption that in an $S$-reallocation $(B_i, x_i)_{i \in S}$ the vector $x \in \mathbb{R}_+^S$ should be non-negative, in fact states that money is scarce. This assumption is also made in Beviá et al. (1999). Observe, that this assumption is only restrictive inasmuch as it places an upper-bound on the amount of money an agent may borrow. Without loss of generality, one may assume this upper-bound to be zero.

Let $(B_i, x_i)_{i \in \mathcal{N}}$ be an $N$-reallocation and let $S \subseteq \mathcal{N}$ be non-empty. An $S$-reallocation $(C_i, y_i)_{i \in S}$ is a weak improvement upon $(B_i, x_i)_{i \in \mathcal{N}}$ if $V_i(C_i) + y_i \geq V_i(B_i) + x_i$ for all $i \in S$ and $V_j(C_j) + y_j > V_j(B_j) + x_j$ for at least one agent $j \in S$. It is a strong improvement
upon \((B_i, x_i)_{i \in N}\) if \(V_i(C_i) + y_i > V_i(B_i) + x_i\) for all \(i \in S\). As usual, an \(N\)-reallocation is \textit{Pareto efficient/weakly Pareto efficient} if coalition \(N\) has no weak improvement/strong improvement. An \(N\)-reallocation is a \textit{strong core reallocation/core reallocation} if it does \textit{not} admit for a coalition \(S\) to have a weak improvement/strong improvement.

Given an exchange economy \(E\) we define, similarly as in Shapley and Shubik (1969), the corresponding TU-game \((N, v_E)\) by

\[
v_E(S) := \max \left\{ \sum_{i \in S} V_i(C_i) \mid (C_i)_{i \in S} \text{ is an } S\text{-redistribution} \right\} \quad \text{for all } S \subseteq N,
\]

i.e., \(v_E(S)\) is the \textit{maximum social welfare} in the sub-economy with only agents in coalition \(S\). Note that the TU-game \((N, v_E)\) is \textit{superadditive}, i.e., \(v_E(S \cup T) \geq v_E(S) + v_E(T)\) whenever \(S \cap T = \emptyset\). Furthermore, since \(V_i(C) \geq V_i(\emptyset) = 0\) for all \(i \in N\) and \(C \subseteq \Omega\), it is a \textit{non-negative} TU-game.

The main goal of this chapter is to extend certain solution concepts known in the theory of TU-games to exchange economies and to establish a two-way correspondence between the concepts in the two models. For instance, we would like to extend core elements of the TU-game into (strong) core reallocations of the economy and vice versa. To achieve this goal, the amounts of money initially held by the agents should be large enough. This is demonstrated in the following example.

**Example.** Let \(E\) be an exchange economy with \(N := \{1, 2, 3\}\) and \(\Omega := \{\alpha, \beta, \gamma\}\). The reservation values \(V_i: 2^\Omega \rightarrow \mathbb{R}_+\) are for each \(i \in N\) the same and given by:

<table>
<thead>
<tr>
<th>agents 1–3</th>
<th>{\alpha}</th>
<th>{\beta}</th>
<th>{\gamma}</th>
<th>{\alpha, \beta}</th>
<th>{\alpha, \gamma}</th>
<th>{\beta, \gamma}</th>
<th>{\alpha, \beta, \gamma}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

Let \(A_1 := \{\alpha\}, A_2 := \{\beta\}\) and \(A_3 := \{\gamma\}\). Although, the initial amounts of money \(m \in \mathbb{R}_+^N\) are not yet specified, it can already be verified that the TU-game \((N, v_E)\) has a non-empty core. Indeed, \(v_E(N) = 10, v_E(S) = 5\) whenever \(|S| = 2\) and \(v_E(S) = 0\) whenever \(|S| \leq 1\).

Now let us discuss the strong core and the core of the economy \(E\) while varying \(m \in \mathbb{R}_+^N\). Assume that \(m_1 := 2\) and \(m_2 = m_3 := 1\). Then it can be verified that the strong core is empty. Indeed, the best candidate would be the \(N\)-reallocation \((\{\alpha, \beta, \gamma\}, 0), (\emptyset, 2), (\emptyset, 2)\). However, it allows for coalition \(\{2, 3\}\) to have a weak improvement, via the \(\{2, 3\}\)-reallocation \((\emptyset, 2), (\{\beta, \gamma\}, 0)\), and therefore it is \textit{not} a strong core reallocation. Observe that this weak improvement of coalition \(\{2, 3\}\) \textit{cannot} be transformed into a strong improvement. This is due to the lack of money of agents 2 and 3. Hence, the \(N\)-reallocation is an element of the core of the economy.

Next, assume that \(m_i := 2\) for all \(i \in N\). In this case the core is empty. Indeed, similar as before, the only possible core candidate is \((\{\alpha, \beta, \gamma\}, 0), (\emptyset, 3), (\emptyset, 3)\). This time however, it allows for coalition \(\{2, 3\}\) to have a strong improvement.

Finally, assume that \(V_i(\{\alpha, \beta, \gamma\}) := 5\) (instead of 10) for all \(i \in N\). Then the TU-game \((N, v_E)\) has an empty core. However, if \(m_1 := 2\) and \(m_2 = m_3 := 1\), then the \(N\)-reallocation \(((\{\alpha, \beta, \gamma\}, 0), (\emptyset, 2), (\emptyset, 2))\) is still a core reallocation.
So, although we assume for money to be scarce, the initial amounts of money held by the agents should be large enough to avoid phenomena like in the example above. Therefore, we give the following abundance condition for the amounts of money agents initially have.

**Definition.** An exchange economy $E$ satisfies the Total Abundance (TA) condition if every coalition $S \subseteq N$ has an $S$-redistribution $(C_i)_{i \in S}$ such that

$$\sum_{i \in S} V_i(C_i) = v_E(S) \quad \text{and} \quad V_i(C_i) \leq V_i(A_i) + m_i \quad \text{for all } i \in S.$$  

Such an $S$-redistribution $(C_i)_{i \in S}$ is said to satisfy the TA-condition for coalition $S$.

Assume that for coalition $S \subseteq N$ to maximize its social welfare agent $i \in S$ should receive $C_i$. The TA-condition stipulates that agent $i \in S$ has enough money to buy $C_i$ for the price $V_i(C_i)$ (the highest price he is willing to pay), when he can sell $A_i$ for the price $V_i(A_i)$ (the lowest price for which he is willing to sell $A_i$). The TA-condition is weaker than abundance conditions that can be found in the literature, namely, $V_i(C) \leq V_i(A_i) + m_i$ or even $V_i(C) \leq m_i$ for all $i \in N$ and $C \subseteq \Omega$ (see e.g., Bikhchandani and Mamer (1997) or Beviá et al. (1999)).

The following lemma states that in an exchange economy satisfying the TA-condition individual rational $N$-reallocations admit a strong improvement whenever they admit a weak improvement. An $N$-reallocation is said to be individual rational whenever each agent receives a bundle which he (weakly) prefers to his initial endowment.

**Lemma 3.2.** Let $E$ be an exchange economy satisfying the TA-condition. Let $(B_i, x_i)_{i \in N}$ be an $N$-reallocation such that $V_i(B_i) + x_i \geq V_i(A_i) + m_i$ for all $i \in N$. If $(B_i, x_i)_{i \in N}$ admits a weak improvement, then it also admits a strong improvement.

**Proof.** Let $(B_i, x_i)_{i \in N}$ be an $N$-reallocation such that $V_i(B_i) + x_i \geq V_i(A_i) + m_i$ for all $i \in N$. Assume that coalition $S \subseteq N$ has a weak improvement upon this $N$-reallocation, say via the $S$-reallocation $(C_i', y_i')_{i \in S}$. Then, by definition,

$$v_E(S) + \sum_{i \in S} m_i \geq \sum_{i \in S} [V_i(C_i') + y_i'] > \sum_{i \in S} [V_i(B_i) + x_i]. \tag{1}$$

Let $(C_i)_{i \in S}$ be an $S$-redistribution which satisfies the TA-condition for coalition $S$. For all $i \in S$ we define $z_i := V_i(B_i) + x_i - V_i(C_i)$. Observe that

$$z_i \geq V_i(A_i) + m_i - V_i(C_i) \geq 0 \quad \text{for all } i \in S.$$

The last inequality follows from the TA-condition. Furthermore, due to inequality (1), we have that $\delta := \frac{1}{|S|} \sum_{i \in S} [m_i - z_i] > 0$. Next, define $y_i := z_i + \delta$ for all $i \in S$. Then $(C_i, y_i)_{i \in S}$ is an $S$-reallocations and a strong improvement upon $(B_i, x_i)_{i \in N}$. \[\square\]

Since, core reallocations are in particular individual rational, Lemma 3.2 tells us that the strong core and the core coincide for exchange economies satisfying the TA-condition. We write $\mathcal{C}(E)$ for the set of (strong) core reallocations in such an economy $E$. Note that Lemma 3.2 is not true if the TA-condition is violated. This is illustrated in the first example of
this section. The next example demonstrates that for an exchange economy satisfying the
TA-condition, the difference between Pareto efficiency and weak Pareto efficiency remains.
The reason is that a weak Pareto efficient reallocation is not necessarily individual rational.

Example. Let $E$ be an exchange economy with $N := \{1, 2\}$, $\Omega := \{\alpha, \beta\}$ and initial endow-
ments $(A_i, m_i)_{i \in N} := ((\alpha, 1), (\beta, 1))$. The reservation values $V_i : 2^\Omega \rightarrow \mathbb{R}_+$ for $i = 1, 2$
are given by:

<table>
<thead>
<tr>
<th>Agent</th>
<th>${\alpha}$</th>
<th>${\beta}$</th>
<th>${\alpha, \beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>7</td>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td>Agent 2</td>
<td>7</td>
<td>9</td>
<td>16</td>
</tr>
</tbody>
</table>

The exchange economy $E$ satisfies the TA-condition, since $v_E(N) = V_1(A_1) + V_2(A_2)$. However, $((\beta, 1), (\alpha, 1))$ is a weak Pareto efficient reallocation, but it is not Pareto efficient. Indeed, it allows for the weak improvement $((\alpha, 2), (\beta, 0))$, but this weak improvement cannot be transformed into a strong improvement.

The following proposition illustrates that every (strong) core reallocation of an economy $E$
which satisfies the TA-condition gives rise to a core element of the TU-game $(N, v_E)$ and
vice versa.

Proposition 3.3. Let $E$ be an exchange economy which satisfies the TA-condition.

(i) If $(B_i, x_i)_{i \in N} \in C(E)$, then $(V_i(B_i) + x_i - m_i)_{i \in N} \in C(v_E)$.

(ii) If $X \in C(v_E)$, then $(B_i, X_i + m_i - V_i(B_i))_{i \in N} \in C(E)$ for some $N$-redistribution $(B_i)_{i \in N}$.

Proof. (i) Let $(B_i, x_i)_{i \in N} \in C(E)$ and define $X_i := V_i(B_i) + x_i - m_i$ for all $i \in N$. Suppose $X(S) < v_E(S)$ for some coalition $S \subseteq N$ (observe that also $S = N$ is one of the possibilities). Let $(C_i)_{i \in S}$ be an $S$-redistribution which satisfies the TA-condition for coalition $S$. Then

$$\sum_{i \in S} [V_i(B_i) + x_i] = X(S) + \sum_{i \in S} m_i < v_E(S) + \sum_{i \in S} m_i = \sum_{i \in S} [V_i(C_i) + m_i].$$

For all $i \in S$ there is a number $y_i \in \mathbb{R}$ such that $V_i(C_i) + y_i = V_i(B_i) + x_i$. Take $i \in S$. Then by the core-condition for coalition $\{i\}$ we have,

$$V_i(B_i) + x_i \geq V_i(A_i) + m_i.$$

Therefore, $y_i = V_i(B_i) + x_i - V_i(C_i) \geq V_i(A_i) + m_i - V_i(C_i) \geq 0$ for all $i \in S$. Define $\delta := \frac{1}{|S|} \sum_{i \in S} [m_i - y_i]$. Since, $\sum_{i \in S} m_i > \sum_{i \in S} y_i$, we conclude that the $S$-reallocation $(C_i, y_i + \delta)_{i \in S}$ is a strong improvement for coalition $S$ upon $(B_i, x_i)_{i \in N}$. This yields the desired contradiction.

(ii) Let $X \in C(v_E)$. Take an $N$-redistribution $(B_i)_{i \in N}$ which satisfies the TA-condition for
coalition $N$ and define $x_i := X_i + m_i - V_i(B_i)$ for all $i \in N$. Observe that $\sum_{i \in N} x_i = \sum_{i \in N} m_i$ and that $x_i \geq V_i(A_i) + m_i - V_i(B_i) \geq 0$ for all $i \in N$. Hence, $(B_i, x_i)_{i \in N}$ is an $N$-reallocation.
Suppose the coalition \( S \subseteq N \) has an improvement upon this \( N \)-reallocation, i.e., there exists an \( S \)-reallocation \( (C_i, y_i)_{i \in S} \) such that
\[
V_i(C_i) + y_i > V_i(B_i) + x_i \quad \text{for all } i \in S.
\]
This means that \( V_i(C_i) + y_i > X_i + m_i \) for all \( i \in S \). Because \( \sum_{i \in S} y_i = \sum_{i \in S} m_i \), this inequality yields, \( v_E(S) \geq \sum_{i \in S} V_i(C_i) > X(S) \). But \( X \in C(v_E) \). Contradiction. \( \square \)

Recall from the first example in this section that if the TA-condition is violated, then Proposition 3.3 is no longer true. However, if an exchange economy \( E \) satisfies the TA-condition, then the corresponding TU-game \( (N, v_E) \) is balanced if and only if the core \( C(E) \) is non-empty. In the next section, we study two other sets of \( N \)-reallocations which can be seen as a substitute of the core when the latter is empty, namely, the reactive and the semireactive bargaining set.

### 3.3 Bargaining sets in exchange economies

In Chapter 2 we investigated the reactive and the semireactive bargaining set for (symmetric) TU-games. These concepts can be extended to exchange economies with indivisible goods and money. To do so, we first need to reformulate the definitions of an objection and of a counter-objection. Recall that for all \( k, \ell \in N \) with \( k \neq \ell \) we denote \( \Gamma_{k\ell} := \{ S \subseteq N \mid k \in S \subseteq N \setminus \{\ell\} \} \).

**Definition.** An objection of agent \( k \) against another agent \( \ell \) with respect to an \( N \)-reallocation \( (B_i, x_i)_{i \in N} \) is a pair \( (P, (C_i, y_i)_{i \in P}) \) with \( P \in \Gamma_{k\ell} \) and \( (C_i, y_i)_{i \in P} \) a \( P \)-reallocation such that \( (C_i, y_i) \succ_i (B_i, x_i) \) for all \( i \in P \).

Given an objection of agent \( k \) against agent \( \ell \), with respect to an \( N \)-reallocation, we can give the definition of a counter-objection.

**Definition.** Given an objection \( (P, (C_i, y_i)_{i \in P}) \) of agent \( k \) against agent \( \ell \) with respect to an \( N \)-reallocation \( (B_i, x_i)_{i \in N} \), a counter-objection of agent \( \ell \) against agent \( k \) is a pair \( (Q, (D_i, z_i)_{i \in Q}) \) with \( Q \in \Gamma_{k\ell} \) and \( (D_i, z_i)_{i \in Q} \) a \( Q \)-reallocation such that
\[
(D_i, z_i) \succeq_i (B_i, x_i) \quad \text{for all } i \in Q \setminus P,
\]
\[
(D_i, z_i) \succeq_i (C_i, y_i) \quad \text{for all } i \in P \cap Q.
\]

If agent \( \ell \) is not able to counter, then the objection \( (P, (C_i, y_i)_{i \in P}) \) of agent \( k \) is justified.

Next, we give the formal definitions of the reactive and the semireactive bargaining set for an exchange economy.

**Definition.** Let \( E \) be an exchange economy. A weakly Pareto efficient \( N \)-reallocation is an element of the reactive bargaining set \( M_r(E) \) if for all agents \( k \) and \( \ell \) in \( N \) there exists a coalition \( Q \in \Gamma_{k\ell} \) such that every objection \( (P, (C_i, y_i)_{i \in P}) \) of \( k \) against \( \ell \), with respect to this \( N \)-reallocation, can be countered via \( (Q, (D_i, z_i)_{i \in Q}) \).
So, if in the reactive bargaining set agent \( k \) can formulate an objection against agent \( \ell \), the latter agent must have a coalition \( Q \in \Gamma_{\ell k} \) which he can use to counter all possible objections of agent \( k \).

**Definition.** Let \( \mathcal{E} \) be an exchange economy. A weakly Pareto efficient \( N \)-reallocation is an element of the semireactive bargaining set \( \mathcal{M}_{sr}(\mathcal{E}) \) if for all agents \( k \) and \( \ell \) in \( N \) and every coalition \( P \in \Gamma_{\ell k} \) there exists a coalition \( Q \in \Gamma_{\ell k} \) such that every objection \( (P, (C_i, y_i)_{i \in P}) \) of \( k \) against \( \ell \), with respect to this \( N \)-reallocation, can be countered via \( (Q, (D_i, z_i)_{i \in Q}) \).

So, if in the semireactive bargaining set agent \( k \) announces to object against agent \( \ell \) via coalition \( P \in \Gamma_{k\ell} \), the latter agent must be able to give a coalition \( Q \in \Gamma_{k\ell} \) which he can use to counter all possible objections of agent \( k \) via this coalition \( P \). Given the definitions of an objection and a counter-objection, one also can extend the definition of the bargaining set (Aumann and Maschler (1964)) to exchange economies. For the sake of completeness, we give the formal definition.

**Definition.** Let \( \mathcal{E} \) be an exchange economy. A weakly Pareto efficient \( N \)-reallocation is an element of the bargaining set \( \mathcal{M}(\mathcal{E}) \) if for all agents \( k \) and \( \ell \) in \( N \) and every objection \( (P, (C_i, y_i)_{i \in P}) \) of \( k \) against \( \ell \) with respect to this \( N \)-reallocation there exists a counter-objection \( (Q, (D_i, z_i)_{i \in Q}) \).

**Remark.** In the above definitions we required an \( N \)-reallocation in the particular bargaining set to be weakly Pareto efficient. If one chose to leave out this requirement, then the initial endowments \( (A_i, m_i)_{i \in N} \) would be an element of every bargaining set, since every \( \ell \in N \) can counter any possible objection against him via the one-coalition \( \{\ell\} \).

Note that like in the theory of TU-games (see Section 2.3) we have for an exchange economy \( \mathcal{E} \) the following inclusions

\[
\mathcal{M}_r(\mathcal{E}) \subseteq \mathcal{M}_{sr}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E}).
\]

### 3.3.1 Existence results

Similarly as for TU-games, an \( N \)-reallocation contained in the core does not admit any objection and therefore the core is contained in the reactive bargaining set. Hence, as long as the core is non-empty, the reactive bargaining set is also non-empty. This subsection proves that for the exchange economies studied in this chapter the reactive bargaining set is non-empty, even if the core is empty. By the inclusions given in the previous subsection, the same statement holds for the semireactive bargaining set and the bargaining set. So, if an exchange economy has an empty core there still exists an \( N \)-reallocation which may be appealing, since it does not allow any justified objection. Therefore, one can think of these bargaining sets as (alternative) solution concepts for exchange economies.

**Theorem 3.4.** Let \( \mathcal{E} := (N, \Omega, (A_i, m_i, V_i)_{i \in N}) \) be an exchange economy which satisfies the TA-condition, then \( \mathcal{M}_r(\mathcal{E}) \neq \emptyset \).

**Proof.** Let \( \mathcal{E} \) be an exchange economy which satisfies the TA-condition and let \( (B_i)_{i \in N} \) be an \( N \)-redistribution which satisfies the TA-condition for coalition \( N \). Observe that due to
superadditivity, the TU-game \( (N, v_E) \) has a non-empty imputation set and therefore \( \mathcal{M}_r(v_E) \) is non-empty (Theorem 2.3). Let \( X \in \mathcal{M}_r(v_E) \) and define

\[
x_i := X_i + m_i - V_i(B_i) \quad \text{for all } i \in N.
\]

Observe that \( \sum_{i \in N} x_i = \sum_{i \in N} m_i \) and because \( X_i \geq v_E(\{i\}) = V_i(A_i) \) for all \( i \in N \) it follows from the TA-condition that \( x_i \geq 0 \) for all \( i \in N \). Hence, \( (B_i, x_i)_{i \in N} \) is an \( N \)-reallocation. It is straightforward to verify that this \( N \)-reallocation is (weakly) Pareto efficient. Moreover, we prove that this \( N \)-reallocation is contained in the reactive bargaining set \( \mathcal{M}_r(E) \).

Let \( k \in N \) be an agent who plans to object against agent \( \ell \in N \). Because \( X \in \mathcal{M}_r(v_E) \) there exists a \( Q \in \Gamma_k \) such that for every (possible) objection \( (P, Y) \) from \( k \) against \( \ell \) there exists a vector \( Z \in \mathbb{R}^Q \) with \( Z(Q) = v_E(Q) \) such that \( (Q, Z) \) is a counter-objectation.

Let \( (P, (C_i, y_i)_{i \in P}) \) be an objection from \( k \) against \( \ell \) with respect to the \( N \)-reallocation \( (B_i, x_i)_{i \in N} \). Define \( Y := V_i(C_i) + y_i - m_i \) for all \( i \in P \). Then

\[
Y(P) = \sum_{i \in P} V_i(C_i) + \sum_{i \in P} y_i - \sum_{i \in P} m_i = \sum_{i \in P} V_i(C_i) \leq v_E(P).
\]

Furthermore,

\[
Y_i = V_i(C_i) + y_i - m_i \geq V_i(B_i) + x_i - m_i = X_i \quad \text{for all } i \in P.
\]

So, \( (P, Y) \) is an objection from \( k \) against \( \ell \) with respect to \( X \in \mathcal{M}_r(v_E) \). Hence, there exists a \( Z \in \mathbb{R}^Q \) such that \( (Q, Z) \) is a counter-objectation. Let \( (D_i)_{i \in Q} \) be a \( Q \)-redistribution which satisfies the TA-condition for coalition \( Q \) and define

\[
z_i := Z_i + m_i - V_i(D_i) \quad \text{for all } i \in Q.
\]

Then \( \sum_{i \in Q} z_i = \sum_{i \in Q} m_i \) and because \( Z_i \geq X_i \geq v_E(\{i\}) = V_i(A_i) \) for all \( i \in Q \) it follows from the TA-condition that \( z_i \geq 0 \) for all \( i \in Q \). Hence, \( (D_i, z_i)_{i \in Q} \) is a \( Q \)-reallocation. Furthermore, since \( (Q, Z) \) is a counter-objectation, we also have that

\[
V_i(D_i) + z_i = Z_i + m_i \geq Y_i + m_i = V_i(C_i) + y_i \quad \text{for all } i \in Q \cap P,
\]

\[
V_i(D_i) + z_i = Z_i + m_i \geq X_i + m_i = V_i(B_i) + x_i \quad \text{for all } i \in Q \setminus P.
\]

Hence, the pair \( (Q, (D_i, z_i)_{i \in Q}) \) is a counter-objectation.

\[ \square \]

**Remark.** In the proof of Theorem 3.4 it is shown that if \( X \in \mathcal{M}_r(v_E) \), then the \( N \)-reallocation \( (B_i, X_i + m_i - V_i(B_i))_{i \in N} \) in which \( (B_i)_{i \in N} \) satisfies the TA-condition for coalition \( N \) is contained in the reactive bargaining set \( \mathcal{M}_r(E) \). The proof of Theorem 3.4 can be suitably modified to obtain the same result for the semireactive bargaining set and for the bargaining set.

So, for exchange economies with quasi-linear utilities and satisfying the TA-condition, each of the bargaining sets is non-empty. But what happens if one of these conditions is violated?
3.3 Bargaining sets in exchange economies

First, we revisit the first example of this chapter to illustrate that if the TA-condition does not hold, then each of the bargaining sets may be empty. Thus, the TA-condition cannot simply be omitted in Theorem 3.4.

**Example.** Let \( \mathcal{E} \) be an exchange economy with \( N := \{1, 2, 3\}, \Omega := \{\alpha, \beta, \gamma\} \) and initial endowments \( (A_i, m_i)_{i \in N} := ((\{\alpha\}, 2), (\{\beta\}, 2), (\{\gamma\}, 2)) \). Again the reservation values \( V_i : 2^\Omega \rightarrow \mathbb{R}_+ \) are for each \( i \in N \) given by:

<table>
<thead>
<tr>
<th>agents 1–3</th>
<th>{\alpha}</th>
<th>{\beta}</th>
<th>{\gamma}</th>
<th>{\alpha, \beta}</th>
<th>{\alpha, \gamma}</th>
<th>{\beta, \gamma}</th>
<th>{\alpha, \beta, \gamma}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

In Section 3.2 it is already demonstrated that \( \mathcal{E} \) has an empty (strong) core. Next, we illustrate that the bargaining set is also empty. One candidate for an \( N \)-reallocation in the bargaining set \( \mathcal{M}(\mathcal{E}) \) would be \( ((\{\alpha, \beta, \gamma\}, 0), (\emptyset, 3), (\emptyset, 3)) \). However, it allows for agent 3 to have a justified objection against agent 1, via the \( \{2, 3\} \)-reallocation \( ((\{\beta, \gamma\}, 0), (\emptyset, 4)) \). Indeed, agent 1 can only counter this objection via coalition \( \{1, 2\} \). But, agents 1 and 2 do not have enough money nor high enough reservation values for their initial goods \( \alpha \) and \( \beta \) to exceed the utility level of \( 15 = U_1((\{\alpha, \beta, \gamma\}, 0) + U_2((\{\beta, \gamma\}, 0) \). This means that agent 1 cannot counter the objection of agent 3 via coalition \( \{1, 2\} \).

If agent 1 would receive the bundle \( ((\{\alpha, \beta\}, 0) \) (instead of \( ((\{\alpha, \beta, \gamma\}, 0) \) and either agent 2 or agent 3 would receive \( \{\gamma\} \), then agent 3 has the same justified objection against agent 1.

Using the symmetry in the reservation values and in the initial endowments, we can conclude that \( \mathcal{M}(\mathcal{E}) \) is empty.

For exchange economies with preference relations that are represented by utility functions which are not quasi-linear, one still can define each of the bargaining sets. However, they may be empty. We demonstrate this by proving that the bargaining set may be empty. So, also the notion of quasi-linearity cannot simply be omitted in Theorem 3.4.

**Example.** Let \( \mathcal{E} \) be an exchange economy with \( N := \{1, 2, 3\}, \Omega := \{\alpha_1, \alpha_2, \alpha_3\}, A_i := \{\alpha_i\} \) and \( m_i := 12 \) for \( i = 1, 2, 3 \). The utility functions \( U_i : 2^{\Omega} \times \mathbb{R}_+ \rightarrow \mathbb{R} \) are:

\[
\begin{align*}
U_1(B, x) &:= 7 \cdot W(B) + x, \\
U_2(B, x) &:= 6 \cdot W(B) + (\frac{1}{10}W(B) + 1) \cdot x, \\
U_3(B, x) &:= 5 \cdot W(B) + x
\end{align*}
\]

in which \( W(B) := 1 \) if \( |B| \geq 2 \) and \( W(B) := 0 \) else. So, the reservation values of every agent do only depend on the number of indivisible goods and each agent has the same reservation value for two or three goods. Furthermore, observe that only the utility function of agent 2 is not quasi-linear.

**Claim 1.** If \((B_i, x_i)_{i \in N} \in \mathcal{M}(\mathcal{E}) \), then \( V_i(B_i) + x_i \geq 12 \) for all \( i \in N \).

**Proof.** Let \((B_i, x_i)_{i \in N} \in \mathcal{M}(\mathcal{E}) \). Suppose \( V_k(B_k) + x_k < 12 \) for some \( k \in N \). By weak Pareto efficiency, there is an agent \( \ell \in N \) such that \( V_\ell(B_\ell) + x_\ell > 12 \). Otherwise the \( N \)-reallocation \( ((\{\alpha_k\}, 12 - \varepsilon), (\{\alpha_\ell\}, 12 + \frac{\varepsilon}{2}), (\{\alpha_j\}, 12 + \frac{\varepsilon}{2})) \) with \( \varepsilon > 0 \) sufficiently small would be a strong improvement upon \((B_i, x_i)_{i \in N} \).
Clearly, the pair \( \{(k), \{x_k\}, 12\} \) is an objection of agent \( k \) against agent \( \ell \). Since agent \( \ell \) strictly prefers the bundle \( (B_\ell, x_\ell) \) to his initial bundle \( \{(x_\ell), 12\} \) he needs coalition \( \{k, \ell\} \) to counter this objection of agent \( k \). Say, he is able to counter via the \( \{k, \ell\} \)-reallocation \( \left((\{x_k\}, 12 - \epsilon), (D_\ell, z_\ell + \frac{\epsilon}{2}), (D_j, z_j + \frac{\epsilon}{2})\right) \) with \( \epsilon > 0 \) sufficiently small, is a strong improvement upon \( (B_i, x_i) \in N \). This contradicts the weak Pareto efficiency of \( (B_i, x_i) \in N \).

Claim 2. If \( (B_i, x_i) \in N \in \mathcal{M}(\mathcal{E}) \), then \( (B_i, x_i) \in N = \left((\Omega, 36 - 2t), (\emptyset, t), (\emptyset, t)\right) \) for some number \( 12 < t \leq 15 \).

Proof. Let \( (B_i, x_i) \in N \in \mathcal{M}(\mathcal{E}) \). By weak Pareto efficiency, it follows that there is an agent \( j \in N \) such that \( |B_j| \geq 2 \). Due to the reservation values of the agents we may assume without loss of generality that \( B_j = \Omega \). Hence, according to Claim 1, we have that \( x_k, x_\ell \geq 12 \) for the two remaining agents \( k, \ell \in N \setminus \{j\} \).

If \( k = 1 \), then the \( N \)-reallocation \( \left((\Omega, x_k - 7 + \frac{\epsilon}{2}), (\emptyset, x_\ell + \frac{\epsilon}{2}), (\emptyset, x_j + 7 - \epsilon)\right) \) with \( \epsilon > 0 \) sufficiently small is a strong improvement upon \( (B_i, x_i) \in N \). This contradicts the weak Pareto efficiency of \( (B_i, x_i) \in N \). So, we can conclude that \( k, \ell \neq 1 \) and thus \( j = 1 \).

Again, due to Claim 1 this implies that \( x_1 \geq 5 \). Furthermore, if \( 12 \leq x_k < x_\ell \), then the pair \( \left(\{1\}, \{(\alpha_1, \alpha_k), 24 - x_k - \epsilon\}, (\emptyset, x_k + \epsilon)\right) \) with \( \epsilon < \min\{x_k - x_j, 24 - x_k - x_j\} \) is an objection of agent \( k \) against agent \( \ell \). Observe that agent \( \ell \) is not able to counter via coalition \( \{\ell\} \). Hence, there exists a \( \{1, \ell\} \)-reallocation \( ((D_1, z_1), (D_\ell, z_\ell)) \) such that

\[
V_1(D_1) + z_1 + V_\ell(D_\ell) + z_\ell \geq V_1(\{\alpha_1, \alpha_k\}) + 24 - x_k - \epsilon + V_\ell(\emptyset) + x_\ell = 31 - x_k - \epsilon + x_\ell > 31.
\]

But, \( V_1(D_1) + z_1 + V_\ell(D_\ell) + z_\ell \leq 7 + 24 = 31 \). So, we can conclude that \( x_2 = x_3 \geq 12 \). It is left to the reader to verify, by using a similar argument as above, that if \( x_\ell = x_2 = x_3 = 12 \), the pair \( \left(\{2, 3\}, \{\emptyset, 12 + \epsilon\}, (\{\alpha_2, \alpha_3\}, 12 - \epsilon)\right) \) with \( \epsilon > 0 \) sufficiently small is a justified objection of agent 2 against agent 1. Hence, so far we can conclude that

\[
(B_i, x_i) \in N = \left((\Omega, 36 - 2t), (\emptyset, t), (\emptyset, t)\right) \text{ with } t > 12.
\]

Recall that due to Claim 1 we have \( x_3 = 36 - 2 \cdot t \geq 5 \). Thus, \( t \leq 15 \frac{1}{2} \). However, if \( t > 15 \), then it can be verified that agent 1 has a justified objection against agent 3, via the \( \{1, 2\} \)-reallocation \( ((\emptyset, 43 - 2 \cdot t + \epsilon), (\{\alpha_1, \alpha_3\}, 2 \cdot t - 19 - \epsilon)) \) with \( \epsilon > 0 \) sufficiently small. So, we obtain that \( 12 < t \leq 15 \). This proves Claim 2.

Until now, we have not used the fact that the utility function of agent 2 is not quasi-linear. For the remaining part this will be important.

Suppose \( (B_i, x_i) \in N \in \mathcal{M}(\mathcal{E}) \). Then, due to Claim 2, we know that

\[
(B_i, x_i) \in N = \left((\Omega, 36 - 2 \cdot t), (\emptyset, t), (\emptyset, t)\right) \text{ for some number } 12 < t \leq 15.
\]

Now, consider the \( \{1, 3\} \)-reallocation
3.3 Bargaining sets in exchange economies

\[
((\{\alpha_1, \alpha_3\}, 36 - 2 \cdot t), (\emptyset, 2 \cdot t - 12))
\]  
(2)

and the \{2, 3\}-reallocation

\[
((\{\alpha_2, \alpha_3\}, \frac{100}{101} [t - 6]), (\emptyset, 24 - \frac{100}{101} [t - 6])).
\]  
(3)

Since \(12 < t \leq 15\), it can be verified that agent 3 strictly improves himself in both reallocations with respect to the bundle \((\emptyset, t)\). Agent 1 and agent 2 obtain in these reallocations bundles which they prefer equally as the bundle \((\Omega, 36 - 2 \cdot t)\) and \((\emptyset, t)\), respectively. So, since the utility functions are continuous in money, both reallocations can be transformed into objections with respect to \((B_i, x_i)_{i \in \mathbb{N}}\). To be more precise, the \{1, 3\}-reallocation in (2) can be transformed into an objection from agent 1 against agent 2 with respect to \((B_i, x_i)_{i \in \mathbb{N}}\), and the \{2, 3\}-reallocation in (3) can be transformed into an objection from agent 2 against agent 1 with respect to \((B_i, x_i)_{i \in \mathbb{N}}\).

To counter this possible objection of agent 2, agent 1 needs coalition \{1, 3\}. In fact, the best attempt to counter is the \{1, 3\}-reallocation in (2). Similarly, agent 2 needs the \{2, 3\}-reallocation in (3) to counter this possible objection of agent 1. Hence, unless

\[2 \cdot t - 12 = 24 - \frac{100}{101} [t - 6],\]

agent 3 strictly prefers one of the two bundles \((\emptyset, 2 \cdot t - 12)\) or \((\emptyset, 24 - \frac{100}{101} [t - 6])\) to the other. This would imply that either the reallocation stated in (2) or the reallocation stated in (3) could be transformed into a justified objection.

Therefore, we can conclude that \(2 \cdot t - 12 = 24 - \frac{100}{101} [t - 6]\) and thus \(t = 4 \frac{4}{101}\). But in this case agent 3 can raise an objection against agent 1 in which he offers agent 2 the bundle \((\{\alpha_2, \alpha_3\}, 9 \frac{447}{151} - \varepsilon)\) (and keeps the bundle \((\emptyset, 14 \frac{447}{151} + \varepsilon)\) for himself) with \(\varepsilon > 0\) sufficiently small. The best agent 1 can offer agent 2, in order to counter this objection, is the bundle \((\emptyset, 16 \frac{8}{151})\). However,

\[
U_2((\alpha_2, \alpha_3), 9 \frac{447}{151} - \varepsilon) = 6 + \frac{100}{101} [9 \frac{447}{151} - \varepsilon] > 16 \frac{8}{151} = U_2((\emptyset, 16 \frac{8}{151})).
\]

So, the objection is justified and the desired contradiction is obtained. Hence, the bargaining set \(\mathcal{M}(\mathcal{E}) = \emptyset\) and as a result \(\mathcal{M}_f(\mathcal{E}) = \mathcal{M}_{sr}(\mathcal{E}) = \mathcal{M}(\mathcal{E}) = \emptyset\).

**Remark.** The utility functions in the example are complete, transitive and satisfy next to strict monotonicity and continuity in money also the archimedean property. Moreover, the indivisible goods are weakly desired by all agents \(i \in \mathbb{N}\). However, the utility function \(U_2 : \mathcal{O} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+\) of agent 2 does not satisfy the property of constant marginal utility for money. So, in fact this property on the preferences cannot simply be omitted for the non-emptiness of the bargaining set in exchange economies.

3.3.2 Bargaining sets and the core

According to the proof of Theorem 3.4 we obtain that an element of the (semi)reactive bargaining set \(\mathcal{M}_{sr}(v_\varepsilon)\) of the TU-game \((\mathbb{N}, v_\varepsilon)\) can be converted into an \(N\)-reallocation in
Case (1). If 

We distinguish between the following two cases: Observe that

the (semi)reactive bargaining set \( M_{sr}(\mathcal{E}) \) of the exchange economy \( \mathcal{E} \). The converse of this statement is also true. To prove this, we first need the following lemma which states that every \( N \)-reallocation in the semireactive bargaining set is individual rational. Observe, that this also implies for the \( N \)-reallocations in the reactive bargaining set to be individual rational.

**Lemma 3.5.** Let \( \mathcal{E} \) be an exchange economy satisfying the TA-condition. If \((B_i, x_i)_{i \in N} \in M_{sr}(\mathcal{E})\), then \( V_i(B_i) + x_i \geq V_i(A_i) + m_i \) for all \( i \in N \).

**Proof (Due to Peter Sudhölter).** Let \((B_i, x_i)_{i \in N} \in M_{sr}(\mathcal{E})\) and define the set

\[ I := \{ i \in N \mid V_i(B_i) + x_i < V_i(A_i) + m_i \}. \]

Suppose \( I \neq \emptyset \) and let \( k \in I \). For every \( P \subseteq N \setminus \{k\} \) we denote

\[ E(P) := v_{\mathcal{E}}(P) - \sum_{i \in P \cap I} V_i(A_i) - \sum_{i \in P \setminus I} [V_i(B_i) + x_i - m_i]. \]

Observe that due to superadditivity of \((N, v_{\mathcal{E}})\) we have \( E(P \cup Q) \geq E(P) + E(Q) \) whenever \( P \cap Q = \emptyset \). Define

\[ \mu := \max \{ E(P) \mid P \subseteq N \setminus \{k\} \} \quad \text{and} \quad \mathcal{P} := \{ P \subseteq N \setminus \{k\} \mid E(P) = \mu \}. \]

Let \( P^* \in \mathcal{P} \) such that \( |P^*| \) is maximal. Observe that \( \mu \geq E(P) \geq 0 \) whenever \( P \subseteq I \setminus \{k\} \).

Therefore, due to maximality of \(|P^*|\), we have \( I \setminus \{k\} \subseteq P^* \). Select \( \varepsilon > 0 \) satisfying

\[ |P^*| \cdot \varepsilon < \min \{ m_k, V_k(A_k) + m_k - V_k(B_k) - x_k \} \]

and let \((C_i)_{i \in P^*}\) be a \( P^* \)-redistribution satisfying the TA-condition for coalition \( P^* \). Let \( y \in R^{P^* \cup \{k\}} \) be defined by

\[ y_i := \begin{cases} V_i(A_i) + m_i + \varepsilon - V_i(C_i) & \text{if } i \in I \setminus \{k\}, \\ V_i(B_i) + x_i + \varepsilon - V_i(C_i) & \text{if } i \in P^* \setminus I, \\ m_k - |P^*| \cdot \varepsilon & \text{if } i = k. \end{cases} \]

Observe that \( y_i \geq 0 \) for all \( i \in P^* \cup \{k\} \) and that

\[ \sum_{i \in P^* \cup \{k\}} y_i = \sum_{i \in I \setminus \{k\}} V_i(A_i) + \sum_{i \in I} m_i + \sum_{i \in P^* \setminus I} [V_i(B_i) + x_i] - \sum_{i \in I} V_i(C_i) \]

\[ = \sum_{i \in I \setminus \{k\}} V_i(A_i) + \sum_{i \in P^* \setminus I} [V_i(B_i) + x_i - m_i] + \sum_{i \in P^* \cup \{k\}} m_i - v_{\mathcal{E}}(P^*) \]

\[ = \sum_{i \in P^* \cup \{k\}} m_i - \mu. \]

We distinguish between the following two cases:

**Case (1).** If \( P^* = \emptyset \), then \( I = \{k\} \) and \( \mu = 0 \). Take \( \ell \in N \setminus \{k\} \). Because the pair \((\{k\}, (A_k, m_k))\) is an objection of agent \( k \) against agent \( \ell \), there exists a counter-objection \((Q, (D_i, z_i)_{i \in Q})\). Since in particular \( Q \subseteq N \setminus \{k\} \) and \( I = \{k\} \) we obtain that
3.3 Bargaining sets in exchange economies

\[ E(Q) = v_E(Q) + \sum_{i \in Q} m_i - \sum_{i \in Q} [V_i(B_i) + x_i] \]
\[ \geq \sum_{i \in Q} [V_i(D_i) + z_i] - \sum_{i \in Q} [V_i(B_i) + x_i] \geq 0 = \mu. \]

Hence, \( E(Q) = \mu. \) But since \( Q \neq \emptyset, \) we have a contradiction to the maximality of \( |P^*|. \)

Case (2). If \( P^* \neq \emptyset, \) then we define for every \( j \in P^*, \)
\[ R_j := \{(A_k, y_k), (C_j, y_j + \mu), (C_i, y_i)_{i \in P \setminus \{j\}}\}. \]

Observe that \( R_j \) is a \( P^* \cup \{k\}\)-reallocations and for every \( j \in P^* \) a strong improvement upon \((B_i, x_i)_{i \in N} \in M_{\text{sr}}(\mathcal{E}). \) So, by weak Pareto efficiency, this yields that \( P^* \cup \{k\} \neq N. \) Let \( \ell \in N \setminus (P^* \cup \{k\}). \) Then agent \( k \) can raise an objection against agent \( \ell. \) Hence, for every objection via coalition \( P^* \cup \{k\} \) there exists a coalition \( Q \in \Gamma_{\ell k} \) which \( \ell \) can use to counter every objection via \( P^* \cup \{k\}. \) In particular this yields \( E(Q) \geq 0. \) Due to the maximality of \( |P^*| \) it follows that \( |P^* \cap Q| \neq \emptyset. \) Take \( j \in P^* \cap Q. \) We prove that the objection \( (P^* \cup \{k\}, R_j) \) cannot be countered via coalition \( Q. \) Suppose that the pair \((Q, (D_i, x_i)_{i \in Q})\) is a counter-objection of agent \( \ell. \) Then it is straightforward to verify that
\[ \sum_{i \in Q} [V_i(D_i) + z_i] > \sum_{i \in Q \setminus I} [V_i(B_i) + x_i] + \sum_{i \in Q \cap I} [V_i(A_i) + m_i] + \mu. \] (4)

Because \( v_E(Q) + \sum_{i \in Q} m_i \geq \sum_{i \in Q} [V_i(D_i) + z_i] \) and \( Q \subseteq N \setminus \{k\}, \) inequality (4) contradicts the definition of \( \mu \). This yields the desired contradiction. \[ \square \]

**Remark.** According to Lemma 3.5 every \( N\)-reallocations in the (semi)reactive bargaining set is individual rational. Due to Lemma 3.2, this implies in particular for the \( N\)-reallocations in the (semi)reactive bargaining set not only to be weakly Pareto efficient, but even to be Pareto efficient.

In the proof of Lemma 3.5 we explicitly used the fact that in the (semi)reactive bargaining the agent who can expect an objection has to give the coalition he uses to counter before the reallocation in the objection is specified. It might be of interest to point out that Lemma 3.5 is not true for the bargaining set.

**Example.** Let \( \mathcal{E} \) be an exchange economy with \( N := \{1, \ldots, 4\} \) and \( \Omega := \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}. \) The initial endowments are \( A_i := \{\alpha_i\}, m_i = 12 \) for \( i \in \{1, 2, 3\} \) and \( A_4 := \{\beta_1, \beta_2, \beta_3\}, m_4 := 2. \) The reservation values \( V_i : 2^\Omega \rightarrow \mathbb{R} \) for all \( i \in N \) are given by:

| agents 1–3 | \{\alpha_\beta\} | \{\alpha_\beta, \alpha_\beta\} | \{\alpha_\beta, \alpha_\beta\} | \{\alpha_\beta, \alpha_\beta\} | \{\alpha_\beta, \alpha_\beta\} |
|---|---|---|---|---|
| \{\alpha_\beta\} | 0 | 1 | 12 | 12 | 1 | 13 |
| \{\alpha_\beta, \alpha_\beta\} | 0 | 0 | 0 | 0 | 4 | 0 |
| agents 4 | \| |

The remaining reservation values follow by taking the minimal worths that respect monotonicity. Observe that \( v_E(N) = V_1(\{\alpha_1, \alpha_2, \alpha_3\}) + V_4(\{\beta_1, \beta_2, \beta_3\}) = 16. \) Hence, there
exists an \(N\)-redistribution satisfying the TA-condition for coalition \(N\). In fact, it can be easily verified that this exchange economy satisfies the TA-condition. Consider the \(N\)-reallocation

\[
(B_i, x_i)_{i \in N} := (\{\alpha_1, \alpha_2, \alpha_3, \beta_1\}, 4), (\{\beta_2\}, 16), (\{\beta_3\}, 16), (\emptyset, 2)).
\]

Claim. \((B_i, x_i)_{i \in N} \in \mathcal{M}(\mathcal{E})\).

Proof. Observe that the utilities of the agents 1, 2 and 3 for the bundles in the proposed \(N\)-reallocation are all equal to 17. Since, all of their reservation values and their initial endowments are also the same, these agents are interchangeable. Therefore, they can be treated similarly.

The \(N\)-reallocation \((B_i, x_i)_{i \in N}\) is (weakly) Pareto efficient. Indeed, using the symmetry, the best attempt to make a weak improvement is via an \(N\)-reallocation

\[
(\{\alpha_1, \alpha_2, \alpha_3\}, \emptyset, \emptyset, \{\beta_1, \beta_2, \beta_3\}).
\]

However, it cannot be extended to a weak improvement. This can be seen as follows. Agent 1 should at least receive (next to \(\{\alpha_1, \alpha_2, \alpha_3\}\)) an amount of money 5 to exceed the worth of 17 = \(V_1(\{\alpha_1, \alpha_2, \alpha_3, \beta_1\}) + 4\). But then there is not enough money left for agents 2 and 3 such that their utilities exceed the worth of 17 = \(V_2(\{\beta_2\}) + 16 = V_3(\{\beta_3\}) + 16\).

It is left to the reader to verify that none of agents 1, 2 and 3 have a justified objection against one another. Observe that agent 4 can counter every possible objection, since \(V_4(A_4) + m_4 > V_4(B_4) + x_4\). So, the only potential justified objection is from agent 4 to, say, agent 1. The best attempt for agent 4 to do so is via coalition \(P := \{2, 3, 4\}\) and the \(P\)-reallocation

\[
((\{\alpha_2, \alpha_3\}, 7), (\emptyset, 19), (\{\beta_1, \beta_2, \beta_3\}, 0)).
\]

However, agent 1 can counter this objection via coalition \(Q := \{1, 2\}\) and the \(Q\)-reallocation

\[
((\{\alpha_1, \alpha_2\}, 5), (\emptyset, 19)).
\]

Hence, \((B_i, x_i)_{i \in N} \in \mathcal{M}(\mathcal{E})\), but agent 4 strictly prefers his initial endowment \((A_4, m_4)\) to the bundle \((B_4, x_4)\).

With Lemma 3.5 we can now prove that every \(N\)-reallocation in the (semi)reactive bargaining set of the exchange economy can be converted into an element in the (semi)reactive bargaining set of its corresponding TU-game. The proof is only given for the semireactive bargaining set, but it can be easily modified to provide the same result for the reactive bargaining set.

**Proposition 3.6.** Let \(\mathcal{E}\) be an exchange economy satisfying the TA-condition. If \((B_i, x_i)_{i \in N} \in \mathcal{M}_{sr}(\mathcal{E})\), then \((V_1(B_i) + x_i - m_i)_{i \in N} \in \mathcal{M}_{sr}(v_{\mathcal{E}})\).

Proof. Let \((B_i, x_i)_{i \in N} \in \mathcal{M}_{sr}(\mathcal{E})\) and define \(X_i := V_i(B_i) + x_i - m_i\) for all \(i \in N\). Then Lemma 3.5 tells us that \(X_i \geq V_i(A_i) = v_\mathcal{E}(\{i\})\) for all \(i \in N\).

Claim. \(X(N) = v_\mathcal{E}(N)\).
3.3 Bargaining sets in exchange economies

Proof. Suppose $\sum_{i \in N} V_i(B_i) < v_\mathcal{E}(N)$. Let $(C_i)_{i \in N}$ be an $N$-redistribution which satisfies the TA-condition for coalition $N$. Define for all $i \in N$,

$$y_i := V_i(B_i) + x_i - V_i(C_i) + \frac{1}{n} \sum_{i \in N} [V_i(C_i) - V_i(B_i)].$$

Because $V_i(B_i) + x_i \geq V_i(A_i) + m_i$ for all $i \in N$ it follows that $y_i \geq 0$ for all $i \in N$. Furthermore, $\frac{1}{n} \sum_{i \in N} [V_i(C_i) - V_i(B_i)] > 0$. Hence, the $N$-redistribution $(C_i, y_i)_{i \in N}$ is a strong improvement upon $(B_i, x_i)_{i \in N}$. This contradicts the (weak) Pareto efficiency of $(B_i, x_i)_{i \in N}$.

So, $X$ is an imputation of $(N, v_\mathcal{E})$. Next, we prove that $X \in M_{sr}(v_\mathcal{E})$.

Assume agent $k \in N$ can raise an objection against an other agent $\ell \in N$ via coalition $P \in \Gamma_{kl}$. Because $(B_i, x_i)_{i \in N} \in M_{sr}(\mathcal{E})$ there exists a coalition $Q \in \Gamma_{\ell k}$ such that for every objection $(P, (C_i, y_i)_{i \in P})$ from $k$ against $\ell$ there exists a $Q$-reallocation $(D_i, z_i)_{i \in Q}$ such that $(Q, (D_i, z_i)_{i \in Q})$ is a counter-objection.

Let $(P, Y)$ be an objection from $k$ against $\ell$ with respect to $X$. Take a $P$-redistribution $(C_i)_{i \in P}$ which satisfies the TA-condition for coalition $P$. Define $\delta := \frac{1}{|P|}[v_\mathcal{E}(P) - Y(P)]$, then $\delta \geq 0$. Furthermore, define

$$y_i := Y_i - V_i(C_i) + m_i + \delta \text{ for all } i \in P.$$ 

Then $\sum_{i \in P} y_i = \sum_{i \in P} m_i$ and because $Y_i > X_i \geq v_\mathcal{E}([i]) = V_i(A_i)$ it follows from the TA-condition that $y_i \geq 0$ for all $i \in P$. We also have that

$$V_i(C_i) + y_i \geq Y_i + m_i > X_i + m_i = V_i(B_i) + x_i \text{ for all } i \in P.$$ 

So, $(P, (C_i, y_i)_{i \in P})$ is an objection from $k$ against $\ell$ with respect to $(B_i, x_i)_{i \in N} \in M_{sr}(\mathcal{E})$. Thus, there exists a $Q$-reallocation $(D_i, z_i)_{i \in Q}$ such that $(Q, (D_i, z_i)_{i \in Q})$ is a counter-objection. Define $Z_i := V_i(D_i) + z_i - m_i$ for all $i \in Q$. Clearly, $Z(Q) = \sum_{i \in Q} V_i(D_i) \leq v_\mathcal{E}(Q)$. Furthermore, since $(Q, (D_i, z_i)_{i \in Q})$ is a counter-objection, we also have that

$$Z_i = V_i(D_i) + z_i - m_i \geq V_i(C_i) + y_i - m_i \geq Y_i \text{ for all } i \in Q \cap P,$$

$$Z_i = V_i(D_i) + z_i - m_i \geq V_i(B_i) + x_i - m_i = X_i \text{ for all } i \in Q \setminus P.$$ 

Hence, the pair $(Q, Z)$ is a counter-objection and thus $X \in M_{sr}(v_\mathcal{E})$. \(\square\)

Remark. In the proof of Proposition 3.6 we explicitly used Lemma 3.5. We already illustrated, by means of an example, that Lemma 3.5 does not hold in case of the bargaining set. Nevertheless, one can still ask whether every element of the bargaining set of an exchange economy $\mathcal{E}$ at least generates an element of the prebargaining set of $(N, v_\mathcal{E})$ (the vector $X \in \mathbb{R}^N$ is in the prebargaining set if $X$ is efficient and it does not allow, like in the bargaining set, any justified objections. However, $X$ is not necessarily individual rational). The example stated directly after Lemma 3.5 answers this question negatively. In this example, the vector $X := (V_i(B_i) + x_i - m_i)_{i \in N} \in \mathbb{R}^N$, derived from the proposed $N$-reallocation $(B_i, x_i)_{i \in N} \in M(\mathcal{E})$, is not an element of the prebargaining set of the TU-game $(N, v_\mathcal{E})$, since $X(N) = \sum_{i \in N} V_i(B_i) = 13 + 1 + 1 = 15 < 16 = v_\mathcal{E}(N)$.
Combining the results of Proposition 3.3 and Proposition 3.6 with the proof of Theorem 3.4 yields the following result.

**Corollary 3.7.** Let \( E := (N, \Omega, (A_i, m_i, V_i)_{i \in N}) \) be an exchange economy which satisfies the TA-condition, then:

\[
\mathcal{M}_{sr}(E) = C(E) \quad \text{if and only if} \quad \mathcal{M}_{sr}(v_E) = C(v_E),
\]
\[
\mathcal{M}_r(E) = C(E) \quad \text{if and only if} \quad \mathcal{M}_r(v_E) = C(v_E).
\]

For an exchange economy satisfying the TA-condition and with four or fewer agents, the (semi)reactive bargaining set and the core coincide whenever the latter is non-empty. This follows immediately by Corollary 3.7 and the result by Solymosi (2002). Recall that the exchange economies studied in this chapter give rise to (non-negative) superadditive TU-games. It may be of interest to point out the fact that also every non-negative and superadditive TU-game generates an exchange economy, since several classes of superadditive balanced TU-games can be found in the literature for which the core and the (semi)reactive bargaining set coincide (see Section 2.1 for an overview).

**Proposition 3.8.** If \( (N, v) \) is a non-negative and superadditive TU-game, then there exists an exchange economy \( E \), satisfying the TA-condition, such that \( v_E = v \).

**Proof.** Let \( (N, v) \) be a non-negative and superadditive TU-game. Define \( \Omega := \{\alpha_1, \ldots, \alpha_n\} \) and \( A_i := \{\alpha_i\} \) for all \( i \in N \). The amounts of money \( m_i \in \mathbb{R}_N^+ \) are not relevant and can be chosen such that the TA-condition will be satisfied. Furthermore, we define for all \( i \in N \) the reservation values \( V_i : 2^\Omega \rightarrow \mathbb{R}_+ \) by

\[
V_i(B) := v(\{j \in N \mid \alpha_j \in B\}) \text{ for all } B \subseteq \Omega.
\]

We prove that \( v(S) = v_E(S) \) for all \( S \subseteq N \). Let \( S \subseteq N \). Observe that any \( S \)-redistribution \( (B_i)_{i \in S} \) obeys,

\[
\sum_{i \in S} V_i(B_i) = \sum_{i \in S} v(\{j \in N \mid \alpha_j \in B_i\}) \leq v(S).
\]

The last inequality follows from the superadditivity of \( (N, v) \) in combination with the fact that the collection of coalitions \( \{\{j \in N \mid \alpha_j \in B_i\}\}_{i \in S} \) is in particular a partitioning of coalition \( S \). Hence, according to this inequality we have, \( v_E(S) \leq v(S) \).

By giving all indivisible goods within the coalition \( S \subseteq N \) to exactly one agent, we also have that \( v_E(S) \geq v(S) \) for all \( S \subseteq N \). 

Clearly, two different exchange economies may give the same non-negative and superadditive TU-game. Neither the number of indivisible goods, nor the initial endowments need to be the same. Nevertheless, a non-negative and superadditive TU-game for which the (semi)reactive bargaining set and the core coincide, generates exchange economies in which the (semi)reactive bargaining set and the core also coincide.
3.4 Price equilibria in exchange economies

According to Proposition 3.8 every non-negative and superadditive TU-game generates an exchange economy with quasi-linear utilities. In this section, we extend this result by proving that every non-negative, superadditive and balanced TU-game generates an exchange economy with a price equilibrium. But before we do so, let us start by repeating the definition of a price equilibrium (see e.g., Arrow and Debreu (1954) or Arrow and Hahn (1991)).

Definition. An \( N \)-reallocation \((B_i, x_i) \in \mathbb{N} \) is a price equilibrium, if there exists a price vector \( p \in \mathbb{R}^\mathbb{N} (p \neq 0) \) with the following two properties:

(i) \( p(B_i) + x_i \leq p(A_i) + m_i \) for all \( i \in \mathbb{N} \), (budget constraints)

(ii) If \( V_i(C) + y > V_i(B_i) + x_i \) for a certain \( C \subseteq \Omega, y \in \mathbb{R} \) and \( i \in \mathbb{N} \), then \( p(C) + y > p(A_i) + m_i \). (maximality conditions)

For all \( C \subseteq \Omega \), we denote \( p(C) := \sum_{\alpha \in \mathbb{C}} p_\alpha \).

If \( p \in \mathbb{R}^\mathbb{N} \) is a price vector, the budget agent \( i \in \mathbb{N} \) has to his disposal equals \( p(A_i) + m_i \). So, if \((B_i, x_i) \in \mathbb{N} \) is a price equilibrium supported by this price vector \( p \in \mathbb{R}^\mathbb{N} \), the budget constraint states that each agent \( i \in \mathbb{N} \) did not spend more money than his total budget to purchase the bundle \((B_i, x_i)\). The maximality condition states that by purchasing the bundle \((B_i, x_i)\), agent \( i \in \mathbb{N} \) receives, according to his preferences, the best bundle he can afford within his budget.

Remark. (i) In the literature the concept of a price equilibrium, as stated above, is also referred to as a market equilibrium, competitive equilibrium or Walrasian equilibrium.

(ii) Since, for all agents \( i \in \mathbb{N} \) the utility functions \( U_i : 2^\mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) are strict monotonic in money, the maximality conditions imply that the budget constraints are in fact equalities.

Next, we prove that an exchange economy with a price equilibrium and satisfying the TA-condition generates a balanced TU-game.

Lemma 3.9. If \( \mathcal{E} \) is an exchange economy which satisfies the TA-condition and has moreover a price equilibrium, then the corresponding TU-game \((\mathbb{N}, v_\mathcal{E})\) is balanced.

Proof. Let \( \mathcal{E} \) be an exchange economy with a price equilibrium and satisfying the TA-condition. Let \((B_i, x_i) \in \mathbb{N} \) be a price equilibrium supported by the price vector \( p \in \mathbb{R}^\mathbb{N} \). Suppose coalition \( S \) has a strong improvement upon \((B_i, x_i) \in \mathbb{N} \). So, there exists an \( S \)-reallocation \((C_i, y_i) \in \mathbb{S} \) such that \( V_i(C_i) + y_i > V_i(B_i) + x_i \) for all \( i \in S \). Hence, by the maximality conditions, this yields that \( p(C_i) + y_i > p(A_i) + m_i \) for all \( i \in S \). But, by taking the sum over \( i \in S \), we obtain a contradiction. Hence, \((B_i, x_i) \in \mathbb{S} \in \mathcal{C}(\mathcal{E}) \).
Since the exchange economy $E$ satisfies the TA-condition, we obtain according to Proposition 3.3(i) that $C(v_E) \neq \emptyset$.

**Remark.** Lemma 3.9 in fact shows that a price equilibrium in an exchange economy $E$ is contained in the core $C(E)$ which is a well-known result. Observe, that by definition, this implies for a price equilibrium to be an element of the reactive bargaining set $\mathcal{M}_r(E)$.

From Lemma 3.9 we can conclude that for an exchange economy satisfying the TA-condition, the notion of balancedness of the corresponding TU-game is necessary for the existence of a price equilibrium. However, it is known that it is not sufficient. Therefore, another condition can be found in the literature, the so-called Social Welfare condition (see e.g., Potters et al. (2002)). This a condition on the reservation values in the economy.

**Definition.** An exchange economy $E$ satisfies the Social Welfare (SW) condition if the linear program (LP)

\[
\text{maximize } \sum_{i \in N} \sum_{C \subseteq \Omega} \mu_i C \cdot V_i(C) \quad \text{subject to:}
\]

\[
mu_i C \geq 0 \quad \text{for all } i \in N \text{ and } C \subseteq \Omega,
\]

\[
\sum_{C \subseteq \Omega} \mu_i C = 1 \quad \text{for all } i \in N,
\]

\[
\sum_{C, \alpha \in C, i \in N} \mu_i C = 1 \quad \text{for all } \alpha \in \Omega
\]

has an integer-valued optimal solution.

A feasible solution of (LP) can be understood as a stochastic redistribution of $\Omega$. Indeed, by Constraint (5) the number $\mu_i C \geq 0$ can be seen as the probability that agent $i$ receives $C$. According to Constraint (6) each indivisible good $\alpha$ is assigned with probability one to an agent. Observe that an integer-valued feasible solution of (LP) corresponds with an $N$-redistribution. So, the SW-condition states that the highest expected social welfare is obtained by an $N$-redistribution (and therefore it can actually be realized).

The following proposition can be found in Potters et al. (2002) (see also Bikhchandani and Mamer (1997)) which states that the SW- and the TA-condition are sufficient for the existence of a price equilibrium.

**Proposition 3.10 (Potters et al. (2002)).** An exchange economy has a price equilibrium if it satisfies the TA-condition and the SW-condition.

**Proof.** Let $E$ be an exchange economy which satisfies the TA- and SW-condition. Let $(B_i)_{i \in N}$ be an $N$-redistribution which satisfies the TA-condition for coalition $N$ and let $(p^*, q^*) \in \mathbb{R}^\Omega \times \mathbb{R}^N$ be any optimal solution of the following linear program (LP$^*$), which is the dual of (LP),

\[
\text{minimize } p(\Omega) + q(N) \quad \text{subject to:}
\]

\[
p(C) + q_i \geq V_i(C) \quad \text{for all } C \subseteq \Omega \text{ and } i \in N.
\]
3.4 Price equilibria in exchange economies

Note that feasible region of (LP) is compact and thus it has an optimal solution. Therefore, due to duality, also (LP*) has an optimal solution. Furthermore, since Constraint (6) is an equality it can be derived from complementary slackness that \( p^* \neq 0 \).

Define \( x_i := p^*(A_i) + m_i - p^*(B_i) \) for all \( i \in N \). According to the SW-condition the integer-valued solution \( \mu_{iC} := 1 \) if \( C = B_i \) and \( \mu_{iC} := 0 \) else, is an optimal solution of (LP). Hence, by complementary slackness, this yields that \( p^*(B_i) + q_i^* = V_i(B_i) \) for all \( i \in N \). Therefore, for all \( i \in N \),

\[
x_i = p^*(A_i) + m_i - p^*(B_i) = p^*(A_i) + q_i^* + m_i - V_i(B_i) \\
\geq V_i(A_i) + m_i - V_i(B_i) \geq 0.
\]

The last inequality follows from the TA-condition. So, \( (B_i, x_i)_{i \in N} \) is an \( N \)-reallocation which clearly satisfies the budget constraints with respect to the price vector \( p^* \in \mathbb{R}^\Omega \). The maximality conditions are also satisfied. Indeed, if \( V_i(C) + y > V_i(B_i) + x_i \) for some \( C \subseteq \Omega \), \( y \geq 0 \) and \( i \in N \), then

\[
p^*(C) + y \geq V_i(C) - q_i^* + y > V_i(B_i) - q_i^* + x_i = p^*(B_i) + x_i = p^*(A_i) + m_i.
\]

Hence, the \( N \)-reallocation \( (B_i, x_i)_{i \in N} \) is a price equilibrium supported by the price vector \( p^* \in \mathbb{R}^\Omega \).

From Lemma 3.9 and Proposition 3.10 it can be verified that an exchange economy, satisfying the TA- and the SW-condition, generates a non-negative, superadditive and balanced TU-game. The converse of this statement is also true, as we now demonstrate.

**Theorem 3.11.** Every non-negative, superadditive and balanced TU-game generates an exchange economy with a price equilibrium.

**Proof.** Let \( (N, v) \) be a superadditive and balanced TU-game which is non-negative. Define the exchange economy \( \mathcal{E} \) with \( \Omega := \{ \alpha(i) \}_{i \in N} \). In this proof it might be useful to consider \( \alpha : N \rightarrow \Omega \) as a bijection. For all \( i \in N \) we define, similarly as in the proof of Proposition 3.8, \( A_i := \{ \alpha(i) \} \) and the reservation values \( V_i : 2^\Omega \rightarrow \mathbb{R}_+ \) by

\[
V_i(C) := V(C) = v(\{ \alpha^{-1}(C) \}) \text{ for all } C \subseteq \Omega.
\]

For all \( C \subseteq \Omega \), we denote \( \alpha^{-1}(C) := \{ j \in N \mid \alpha(j) \in C \} \). The amounts of money \( m \in \mathbb{R}_+^N \) in \( \mathcal{E} \) are for the moment not relevant. According to Proposition 3.8 this economy \( \mathcal{E} \) generates the TU-game \( (N, v) \), i.e., \( v(\mathcal{E}) = v(S) \) for all \( S \subseteq \Omega \). Next, we illustrate that \( \mathcal{E} \) satisfies the SW-condition.

Let \( \{ \mu_{iC} \}_{C \subseteq \Omega, i \in N} \) be a feasible solution of (LP). Define for all \( S \subseteq N \),

\[
\Lambda_S := \sum_{i \in N} \mu_{i\alpha(S)}.
\]

Then \( \Lambda_S \geq 0 \) for all \( S \subseteq N \). Furthermore, by Constraint (6) of (LP) we have for all \( j \in N \) that
\[ \sum_{S \subseteq N} \lambda_S = \sum_{S \subseteq N} \sum_{i \in N} \mu_{i \alpha(S)} = \sum_{C : \alpha(i) \in C \subseteq N} \mu_C = 1. \]

Hence, \( \{ \lambda_S \}_{S \subseteq N} \) is a non-negative solution of the equation \( \sum_{S \subseteq N} \lambda_S \cdot e_S = e_N \). Balancedness of \( \langle N, v \rangle \) (see Section 2.2) tells us that
\[
\sum_{C \subseteq \Omega} \sum_{i \in N} \mu_C \cdot V(C) = \sum_{S \subseteq N} \sum_{i \in N} \mu_{i \alpha(S)} \cdot V(\alpha(S)) = \sum_{S \subseteq N} \lambda_S \cdot v(S) \leq v(N) = V(\Omega).
\]

As a result, we obtain that (LP) has an integer-valued optimal solution. Indeed, given a fixed agent \( i^* \in N \), the feasible solution \( \mu_C := 1 \) if \( (i, C) = (i^*, \Omega) \) and \( \mu_C := 0 \) else, maximizes social welfare of \( V(\Omega) \). Hence, the highest expected social welfare is obtained by an \( N \)-redistribution and thus the SW-condition is satisfied.

Finally, by setting the initial amounts of money \( m_i \) for all \( i \in N \) large enough such that the TA-condition is satisfied (for instance, define \( m_i := v(N) - v(\{ i \}) \) for all \( i \in N \)) it follows, by Proposition 3.10, that the economy \( E \) has a price equilibrium.

Let us have a look at the following example which is taken from Beviá et al. (1999).

**Example (Beviá et al. (1999)).** Let \( E \) be an exchange economy with \( N := \{1, 2, 3\} \), \( \Omega := \{\alpha, \beta, \gamma\} \) and \( (A_i, m_i)_{i \in N} := ((\{\alpha\}, 6), (\{\beta\}, 3), (\{\gamma\}, 1)) \). The reservation values \( V_i : 2^\Omega \rightarrow \mathbb{R}_+ \) for \( i \in N \) are given by:

<table>
<thead>
<tr>
<th></th>
<th>{\alpha}</th>
<th>{\beta}</th>
<th>{\gamma}</th>
<th>{\alpha, \beta}</th>
<th>{\alpha, \gamma}</th>
<th>{\beta, \gamma}</th>
<th>{\alpha, \beta, \gamma}</th>
</tr>
</thead>
<tbody>
<tr>
<td>agent 1</td>
<td>10</td>
<td>8</td>
<td>2</td>
<td>13</td>
<td>11</td>
<td>9</td>
<td>14</td>
</tr>
<tr>
<td>agent 2</td>
<td>8</td>
<td>5</td>
<td>10</td>
<td>13</td>
<td>14</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>agent 3</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>2</td>
<td>9</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

It is left to the reader to verify that \( E \) satisfies the TA-condition. Beviá et al. prove that the unique \( N \)-redistribution \( (\beta, \alpha, \gamma) \) maximizing social welfare of 24 is not supported by a price vector. The reason is that the stochastic redistribution \( \mu_{1\alpha} = \mu_{1\beta} := \frac{1}{2}, \mu_{2\alpha} = \mu_{2\beta} := \frac{1}{2} \) and \( \mu_{3\gamma} = \mu_{3\delta} := \frac{1}{2} \) obtains an expected social welfare of \( 24 \frac{1}{2} \). So, this economy has no price equilibrium.

Nevertheless, the corresponding TU-game \( \langle N, v_E \rangle \) is balanced. So indeed, balancedness is not sufficient for the existence of a price equilibrium. However, the balanced TU-game \( \langle N, v_E \rangle \) is also generated by the exchange economy \( E' \) with \( N := \{1, 2, 3\} \), \( \Omega := \{\alpha, \beta, \gamma\} \), initial endowments \( (A_i, m_i)_{i \in N} := ((\{\alpha\}, 14), (\{\beta\}, 19), (\{\gamma\}, 16)) \) and the reservation values \( V'_i : 2^\Omega \rightarrow \mathbb{R}_+ \) for all \( i \in N \) given by:

<table>
<thead>
<tr>
<th></th>
<th>{\alpha}</th>
<th>{\beta}</th>
<th>{\gamma}</th>
<th>{\alpha, \beta}</th>
<th>{\alpha, \gamma}</th>
<th>{\beta, \gamma}</th>
<th>{\alpha, \beta, \gamma}</th>
</tr>
</thead>
<tbody>
<tr>
<td>agents 1–3</td>
<td>10</td>
<td>5</td>
<td>8</td>
<td>16</td>
<td>18</td>
<td>13</td>
<td>24</td>
</tr>
</tbody>
</table>

According to Theorem 3.11 the exchange economy \( E' \) has a price equilibrium. Indeed, the \( N \)-reallocator \( ((\{\alpha, \beta\}, 8), (\{\gamma\}, 17), (\emptyset, 24)) \) supported by the price vector \( p := (10, 6, 8) \) is a price equilibrium in \( E' \).
Hence, the economy $E$ has no price equilibrium while the economy $E'$ yielding the same TU-game does have a price equilibrium. This illustrates that the concept of price equilibrium is not a game-theoretical solution concept, i.e., it is a solution concept for exchange economies, not for the corresponding TU-games.

**Remark.** It is known that the phenomenon described in the previous example also occurs in case of the Owen set (Owen (1975)) as a solution rule for linear production processes. These processes give rise to totally balanced TU-games, nevertheless the Owen sets of two linear production processes generating the same TU-game may be different (see Example 2.7 in Gellekom, Potters, Reijnierse, Tijs and Engel (2000)).

If one chose $(A_i)_{i \in N} := (\gamma, \beta, \alpha)$ as the initial $N$-distribution in the (original) exchange economy $E$ of Beviá et al. (1999) (see the previous example), then the corresponding TU-game $\langle N, v_E \rangle$ is not balanced. Because in this case we obtain the strict inequality

$$\frac{1}{2}v_E(\{1, 2\}) + \frac{1}{2}v_E(\{1, 3\}) + \frac{1}{2}v_E(\{2, 3\}) = \frac{1}{2}[18 + 18 + 13] = 24 \frac{1}{2} > 24 = v_E(N).$$

This illustrates that balancedness of the TU-game $\langle N, v_E \rangle$ also depends on the initial $N$-distribution $(A_i)_{i \in N}$ in the exchange economy $E$. Nevertheless, if $E$ satisfies the SW-condition, then for every initial $N$-redistribution the corresponding TU-game $\langle N, v_E \rangle$ is balanced. This statement follows immediately from Lemma 3.9 and Proposition 3.10. So, the SW-condition is *sufficient* for the corresponding TU-game to be balanced, but the following simple example illustrates that it is *not* a necessary condition.

**Example.** Let $E$ be an exchange economy with $N := \{1, 2\}$, $\Omega := \{\alpha, \beta\}$ and the reservation values $V_i : 2^\Omega \rightarrow \mathbb{R}_+$ for $i = 1, 2$ given by:

<table>
<thead>
<tr>
<th></th>
<th>${\alpha}$</th>
<th>${\beta}$</th>
<th>${\alpha, \beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>agent 1</td>
<td>5</td>
<td>7</td>
<td>14</td>
</tr>
<tr>
<td>agent 2</td>
<td>8</td>
<td>9</td>
<td>13</td>
</tr>
</tbody>
</table>

Because $E$ has two agents, we may conclude from the superadditivity that the TU-game $\langle N, v_E \rangle$ is balanced, no matter which initial endowments is chosen. If we set in this example the initial endowments as $(A_1, m_1) := (\{\alpha\}, 3)$ and $(A_2, m_2) := (\{\beta\}, 3)$, then the exchange economy $E$ satisfies the TA-condition and thus, according Proposition 3.3(ii), its (strong) core is non-empty. However, $v_E(N) = 15$ and the stochastic redistribution $\mu_{1\beta3} = \mu_{1\beta} := \frac{1}{2}$ and $\mu_{2\alpha} = \mu_{2\beta} := \frac{1}{4}$ obtains a expected social welfare of $15 \frac{1}{4}$. Hence, the SW-condition is not satisfied, but for any initial endowments the TU-game $\langle N, v_E \rangle$ is balanced.
4 Envy-Freeness and Pareto Efficiency

4.1 Introduction

The subject of this chapter is the existence of envy-free allocations and the compatibility of envy-freeness and Pareto efficiency in economies with finitely many indivisible goods and one perfectly divisible good (money). Most of the results of the chapter are taken from Meertens, Potters and Reijnierse (2000) and from Meertens, Potters and Reijnierse (2002).

The main problem studied in this chapter is to allocate in a fair way a number of indivisible goods (e.g., houses, cars) among a group of people, called agents, when monetary compensations are allowed. The main difference with the economies studied in the previous chapter is that in the present chapter the agents do not have initial endowments, i.e., there is no a priori allocation of the indivisible goods and money. So in this chapter, we are not dealing with the problem how a number of agents should reallocate their initial endowments in an optimal way. Instead, we try to find ‘from scratch’ a distribution of the indivisible goods along with monetary compensations, such that we obtain an allocation considered to be fair by all agents. Like in Chapter 3 the agents have preferences on consumption bundles consisting of a number of indivisible goods and an amount of money. However, in this chapter we investigate more general preferences than in the previous one. The preferences are assumed to be complete, transitive and continuous binary relations on the set of consumption bundles under the two additional assumptions that ‘more money is better than less money’ and that ‘large amounts of money can change preferences’. Again, we call these properties monotonicity in money and the archimedean property, respectively. However, in contrast with Chapter 3, we do not assume for the marginal utility of money to be constant nor for the indivisible goods to be (weakly) desired by all agents.

The first normative concept, studied in this chapter, is envy-freeness. An allocation is envy-
free if no agent prefers the consumption bundle of an other agent to his own bundle. This concept originates from a paper by Foley (1967) (see also Varian (1974) and Maskin (1987)). In fact, it is an elaboration of the ‘fairness’ concept (every agent gets a fair part of the whole), because there are more situations in which envy-freeness makes sense and, quite often, envy-freeness implies fairness when both concepts make sense (‘if you think you get the best part, you have no reason to complain’). The second normative concept, studied in this chapter, is envy-freeness and Pareto efficiency. An allocation is Pareto efficient if no agent can improve himself without hurting the interests of the other agents.

In case of an economy with only divisible goods, one way to prove the existence of envy-free allocations is by extending the economy with initial endowments, i.e., every agent obtains one \( n \)-th of the total quantity of each good, \( n \) being the number of agents. A price equilibrium in such an economy is envy-free and Pareto efficient. Envy-freeness follows from the fact that each agent has the same budget, whatever the price vector may be, and thus they can afford the same bundles. Indeed, as an agent can afford the bundle of any other agent and he has already the best consumption bundle among the bundles he can afford, there will be no envy. Pareto efficiency follows from the First Welfare Theorem (see e.g., Arrow (1951) or Arrow and Hahn (1991)). In an economy with only indivisible goods the initial allocation ‘one \( n \)-th of each of the goods’ makes no sense. However, if one defines the budget of each agent as ‘one \( n \)-th of the price of all the indivisible goods together’, the same reasoning can be followed. Also in this case a price equilibrium is envy-free and Pareto efficient. Observe however, that a price equilibrium places stronger restrictions than envy-freeness. Indeed, in a price equilibrium the bundle assigned to an agent is the best in the set of all bundles he can afford, but in an envy-free allocation each agent only compares his bundle with the \( n - 1 \) bundles assigned to the other agents. Bikhchandani and Mamer (1997) contain necessary and sufficient conditions for the existence of price equilibria if the preference relations can be represented by utility functions which are quasi-linear (see also Section 3.4). However, in this chapter we investigate the existence of envy-free allocations in case of more general preference relations.

This chapter contains a proof for the existence of an envy-free allocation under the rather mild condition that the preferences are complete, transitive and continuous binary relations on the set of consumption bundles which satisfy monotonicity in money and the archimedean property. We start by illustrating that these preference relations can be represented by continuous utility functions. The existence of an envy-free allocation is first proved for the case that the preferences are represented by utility functions which are kink functions, i.e., the utility functions are piecewise-linear. Using this result and a basic result from calculus which states that every continuous function can be approximated uniformly by kink functions, the existence of an envy-free allocation in economies with indivisible goods and money can be guaranteed.

Furthermore in this chapter, the existence of envy-free and Pareto efficient allocations is studied. Svensson (1983) proves that every envy-free allocation is automatically Pareto efficient under the condition that the agents receive exactly one item of the indivisible goods and the utilities are quasi-linear. This result has been extended by Alkan, Demange and Gale (1991) to the case of more general utilities. From these results many people got the idea that envy-
freeness and Pareto efficiency are closely related. In Subsection 4.4.1 however, we give an example in which the set of envy-free allocations and the set of Pareto efficient allocations are both non-empty but disjoint. This is a phenomenon that can occur as soon as agents are allowed to obtain more than one indivisible good and the utilities are not quasi-linear. In the case that the utilities are quasi-linear the set of envy-free allocations and the set of Pareto efficient allocations are not disjoint. However, also in this case not every envy-free allocation is automatically Pareto efficient.

The existence results for envy-free allocations or envy-free and Pareto efficient allocations in the literature put more restrictive conditions on the preferences or on the set of agents and indivisible goods. Mixtures of the following restrictions can be found in the literature:

- The number of indivisible goods is at most equal to the number of agents (Svensson (1983)),
- The indivisible goods have a non-negative marginal value, i.e., the goods are desired by the agents (Quinzii (1994), Gale (1984)),
- The appreciation for the indivisible goods and money is separable,
- The appreciation for money is linear and the same for all agents, i.e., quasi-linear utilities (Klijn (2000), Haake, Raith and Su (2002)).

None of these restrictions are made in this chapter. Only the following conditions are kept:

(i) The marginal value of money is positive (‘more money is better than less money’),

(ii) Preferences are continuous in money,

(iii) The preference of a commodity bundle compared with a commodity bundle which contains no indivisible good can be changed by adding or subtracting (large) amounts of money (‘money matters’).

Properties like (ii) and (iii) are necessary for the existence of envy-free allocations as is shown by examples. Furthermore, we illustrate that the preferences satisfying Properties (i)–(iii) in particular satisfy Condition (2) in Alkan et al. (1991):

(2) For each agent and each commodity bundle there exists a unique amount of money that is deemed equivalent to the commodity bundle.

As the existence result of Alkan et al. (1991) is the closest to the results in this chapter, it might be a good idea to point out the main differences:

(a) The preference relations of Alkan et al. are only defined for commodity bundles \((B, x)\) with \(|B| \leq 1\) and \(x \in \mathbb{R}\),
(b) The agents are only interested in one item of the indivisible goods.

(c) Essentially, the number of agents and the number of indivisible goods are the same.

(d) There is also an amount of money $M$ to divide.

So, in the setting of Alkan et al. the number of agents and the number of indivisible goods need to be the same (see point (c)). However, if these numbers are different they change the situation into a situation with an equal number of agents and indivisible goods. If the number of indivisible goods is smaller than the number of agents, some ‘worthless pieces of paper’ are added to the indivisible goods to obtain equality. If, however, the number of indivisible goods is larger than the number of agents, ‘fictive agents’ are added. These fictive agents appreciate a consumption bundle $(B, x)$ only because of the money. Note that the resulting allocation assigns only a part of the indivisible goods to the ‘real’ agents and that the indivisible goods assigned to fictive agents are in fact not allocated. To give an example, suppose the head of a department has to allocate three tasks, say, $\alpha$, $\beta$ and $\gamma$ among the two members of the department. Suppose there is a budget of 10 units to compensate the agents for the performance of the tasks. To keep things simple we assume that the preferences are represented by the following utility functions:

<table>
<thead>
<tr>
<th>Agent</th>
<th>${\alpha}$</th>
<th>${\beta}$</th>
<th>${\gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-2 + x$</td>
<td>$-3 + x$</td>
<td>$-10 + x$</td>
</tr>
<tr>
<td>2</td>
<td>$-3 + x$</td>
<td>$-2 + x$</td>
<td>$-10 + x$</td>
</tr>
</tbody>
</table>

So, both agents agree that the tasks $\alpha$ and $\beta$ are relatively easy and that the task $\gamma$ is hard to do. Alkan et al. will come with an envy-free allocation $(\{\alpha\}, x) \mapsto 1$, $(\{\beta\}, y) \mapsto 2$ and the fictive agent 3 obtains the difficult task $\gamma$ (he does not mind which task he gets) and a payoff $z$, for some numbers $x, y, z \in \mathbb{R}$. To be envy-free, one must have $x \geq y - 1$, $y \geq x - 1$ and $z \geq \max\{x, y\}$ (since the fictive agent 3 appreciates a consumption bundle only because of the money). Finally, due to feasibility, one must also have $x + y = 10$. So, the whole budget is spent and only the easy tasks are actually executed. A similar example can be formulated in the case of students who rent a house collectively that has more rooms than there are students. In this case some of the rooms remain empty if the setting of Alkan et al. is chosen. One can imagine, however, that some student likes to have an additional room and is willing to pay a larger part of the rent in return.

If there is also an amount of money $M$ to divide in the economy (see point (d)), one can start with an equal distribution of $M$ among the agents and replace for each agent his preference relation $\preceq_1$ with a new preference relation $\preceq_+^1$ such that

$$(B, x) \preceq_+^1 (C, y) \text{ if and only if } (B, x + \frac{M}{n}) \preceq_1 (C, y + \frac{M}{n}).$$

Therewith the economy has been transformed into an economy in which $M = 0$. So, one may assume without loss of generality that $M = 0$.  

### 4.2 Preliminaries

An economy with indivisible goods and money, studied in this chapter, can be described by the tuple \((N, \Omega, (\preceq_{i})_{i \in N})\). Here, \(N\) denotes the set of agents, with \(n := |N| \geq 2\) and \(\Omega\) denotes a finite set of indivisible goods, with \(|\Omega| \geq 1\). Each agent \(i \in N\) is endowed with a preference relation \(\preceq_{i}\) on the set \(2^{\Omega} \times \mathbb{R}\), i.e., the set of consumption bundles \((B, x)\). The coordinate \(x \in \mathbb{R}\) measures the positive or negative quantity of a perfectly divisible good that is referred to as money. The preference relations that are considered in these economies have the following properties:

1. \(\preceq\) is a complete, transitive binary relation on \(2^{\Omega} \times \mathbb{R}\),
2. for all \(B \subseteq \Omega\), if \(x > y\), then \((B, x) \succ (B, y)\) (strict monotonicity in money),
3. for all consumption bundles \((B, x)\) and \((C, y)\) with \((B, x) \succ (C, y)\), there exists a strictly positive number \(\delta\) such that \((B, x - \delta) \succ (C, y)\) and \((B, x) \succ (C, y + \delta)\) (continuity in money),
4. for all consumption bundles \((B, x)\) there exists a positive number \(\Delta\) such that \((B, x) \preceq (\emptyset, \Delta)\) and \((B, x) \succeq (\emptyset, -\Delta)\) (archimedean property).

The set of all preference relations satisfying the Properties (i)–(iv) is denoted by \(\mathcal{R}\). Observe that the preference relations in \(\mathcal{R}\) are more general than the preference relations in Chapter 3.

The main goal of the chapter is to find allocations of the indivisible goods along with monetary compensations which all agents consider to be fair. Therefore, we introduce the following definitions in this context.

**Definition.** An \(n\)-partition of \(\Omega\) is a collection \(\{B_{k}\}_{1 \leq k \leq n}\) with \(B_{k} \subseteq \Omega\) for all \(1 \leq k \leq n\), \(\bigcup_{k=1}^{n} B_{k} = \Omega\) and \(B_{k} \cap B_{\ell} = \emptyset\) whenever \(k \neq \ell\). We allow that \(B_{k} = \emptyset\) for some \(1 \leq k \leq n\). Let \(\{B_{k}\}_{1 \leq k \leq n}\) be an \(n\)-partition of \(\Omega\), \(\pi : N \rightarrow \{1, \ldots, n\}\) a bijection and \(x \in \mathbb{R}^{n}\) a money distribution, i.e., \(\sum_{k=1}^{n} x_{k} = 0\). Then the list \((B_{\pi(i)}, x_{\pi(i)})_{i \in N}\) is an allocation. This means that in this allocation the bundle \((B_{\pi(i)}, x_{\pi(i)})\) is assigned to agent \(i\) by the bijection \(\pi\).

Our first normative concept for an allocation to be fair is no-envy. An allocation is envy-free if no agent strictly prefers the bundle of anyone else to his own. So, in such an allocation each agent receives according to his preference relation the best bundle within this allocation. In Section 4.3, we give a more formal definition of this normative concept. The following example shows that the Properties (iii) and (iv) of a preference relation cannot simply be omitted for the existence of an envy-free allocation.

**Example.** Let \((\{1, 2\}, \{\alpha\}, (\preceq_{i})_{i=1,2}\) be an economy with two agents and one indivisible good.

(i) Assume that both agents have the same preference relation which is strictly monotonic in money and satisfies the following two additional assumptions:
\[
\begin{align*}
(\{\alpha\}, x) & \succ (\emptyset, x) \text{ for all } x \in \mathbb{R}, \\
(\emptyset, x) & \succ (\{\alpha\}, y) \text{ if } x > y.
\end{align*}
\]
This preference relation is not continuous. Indeed, given the situation \((\{\alpha\}, x) \succ (\emptyset, x)\) for some \(x \in \mathbb{R}\), then by subtracting or adding a small amount of money, this preference will be reversed, according to (2). Furthermore, there is no envy-free allocation in this economy. Because, given the allocation \((\emptyset, x), (\{\alpha\}, y)\) for some numbers \(x, y \in \mathbb{R}\), to avoid envy we must have \(x \leq y\) according to (2). However, in this case we have according to strict monotonicity in money and (1) that \((\emptyset, x) \preceq (\emptyset, y) \prec (\{\alpha\}, y)\). So, to avoid envy both agents must get the same amount of money, but then the agent getting the good \(\alpha\) will be envied by the agent not-getting \(\alpha\).

(ii) Assume that both agents have again the same preference relation which is this time strictly monotonic in money and satisfies the following assumption:

\[(\{\alpha\}, x) \succ (\emptyset, y) \text{ for all } x, y \in \mathbb{R}.
\]

It follows immediately, that this preference relation does not have the archimedean property, since the good \(\alpha\) cannot be compensated by any amount of money. Furthermore, there is no envy-free allocation in this economy, since the agent getting the good \(\alpha\) will clearly be envied by the agent not-getting \(\alpha\), whatever he will pay to the latter agent.

Remark. Both examples have a lexicographic flavor. In example (i) money is all important and only when the amounts of money are the same, getting the good \(\alpha\) plays a role. In example (ii) getting the good \(\alpha\) is all important and cannot be compensated by any amount of money.

Later on in this chapter, in Section 4.4, we consider a second concept for allocating the indivisible goods, namely, the concept of no-envy and Pareto efficiency. Recall from Chapter 3 that an allocation is Pareto efficient if no agent can improve himself without hurting the other agents. In Section 4.4 we repeat the formal definition.

This section ends with providing a canonical way to represent each preference relation in \(\mathcal{R}\) by a so-called utility function. In fact, we prove that if the preference relation of an agent is an element of \(\mathcal{R}\), then for each consumption bundle there exists a unique amount of money which is deemed equivalent to this consumption bundle, according to the preference of this agent (this is Condition (2) in Alkan et al. (1991)).

**Proposition 4.1.** For each preference relation \(\preceq \in \mathcal{R}\) there exists a map \(V: 2^\Omega \rightarrow \mathbb{R}\) and a strictly increasing continuous function \(f_B: \mathbb{R} \rightarrow \mathbb{R}\) for all \(B \subseteq \Omega\) with the following properties:

(i) \(V(\emptyset) = 0, \ f_\emptyset(x) = x \) for all \(x \in \mathbb{R}\),

(ii) \(f_B(0) = 0 \) for all \(B \subseteq \Omega\),

(iii) \((B, x) \preceq (C, y)\) if and only if \(V(B) + f_B(x) \leq V(C) + f_C(y)\).

The map \(V\) and the functions \(f_B\) are uniquely determined by \(\preceq\) and the properties (i)–(iii).

**Proof.** We start by proving that
4.3 Envy-free allocations

for every $B \subseteq \Omega$ and every $x \in \mathbb{R}$ there exists exactly one real number $u_B(x) \in \mathbb{R}$ such that $(B, x) \sim (\emptyset, u_B(x))$.

Let $B \subseteq \Omega$ and $x \in \mathbb{R}$. Define $K := \{ y \in \mathbb{R} | (B, x) \preceq (\emptyset, y) \}$ and $G := \{ y \in \mathbb{R} | (B, x) \succeq (\emptyset, y) \}$. Then $K$ and $G$ are non-empty and closed, according to the archimedean property and continuity in money, respectively. By completeness of $\preceq$ we have $K \cup G = \mathbb{R}$. Hence, $K \cap G \neq \emptyset$.

Similarly as in the proof of Proposition 3.1, by using transitivity of $\preceq$ and strict monotonicity in money, it follows that $|K \cap G| = 1$. Hence, for every $B \subseteq \Omega$ and $x \in \mathbb{R}$, there exists a unique number $u_B(x) \in \mathbb{R}$ such that $(B, x) \sim (\emptyset, u_B(x))$.

Accordingly, define $V(B) := u_B(0)$ for all $B \subseteq \Omega$ and $f_B(x) = u_B(x) - V(B)$ for all $B \subseteq \Omega$ and $x \in \mathbb{R}$. Then $V : 2^\Omega \rightarrow \mathbb{R}$ and $f_B : \mathbb{R} \rightarrow \mathbb{R}$ are uniquely determined and it is straightforward to verify that these maps satisfy the properties (i)–(iii) mentioned in the proposition.

**Remark.** Conversely, if $\preceq$ is a binary relation on $2^\Omega \times \mathbb{R}$ that can be represented by the utility function

$$U(B, x) := V(B) + f_B(x)$$

in which $f_B : \mathbb{R} \rightarrow \mathbb{R}$ are continuous and strictly increasing functions with $f_\emptyset(x) = x$ for all $x \in \mathbb{R}$, then $\preceq$ belongs to $\mathcal{R}$.

From here, an economy $(N, \Omega, (\preceq_i)_{i \in N})$ is considered in which for all $i \in N$ the preference relation $\preceq_i$ is represented by a utility function $U_i(B, x) = V_i(B) + f_{iB}(x)$ in which $V_i(B)$ are real numbers and $f_{iB} : \mathbb{R} \rightarrow \mathbb{R}$ are strictly increasing and continuous functions for all $B \subseteq \Omega$ with the additional convention that $V_i(\emptyset) = 0$ and $f_{i\emptyset}(x) = x$ for all $x \in \mathbb{R}$.

4.3 Envy-free allocations

This section contains a proof for the existence of an envy-free allocation in the economies $(N, \Omega, (\preceq_i)_{i \in N})$ with $\preceq_i \in \mathcal{R}$ for all $i \in N$. Let us start by giving the formal definition of such an allocation.

**Definition (Foley (1967)).** An allocation $(B_{\pi(i)}, x_{\pi(i)})_{i \in N}$ is envy-free if

$$U_i(B_{\pi(i)}, x_{\pi(i)}) \geq U_i(B_{\pi(j)}, x_{\pi(j)}) \text{ for all } i, j \in N.$$

So, in an envy-free allocation each agent $i \in N$ prefers his bundle $(B_{\pi(i)}, x_{\pi(i)})$ over all other bundles in the allocation.

Next, we define the envy-measure of an allocation. Given an allocation this function measures 'how much envy' there is among the agents expressed in the utility functions of the agents. We write $\Pi$ as the family of all bijections $\pi : N \rightarrow \{1, \ldots, n\}$ and we denote $\mathcal{M} := \{ x \in \mathbb{R}^n \mid \sum_{k=1}^n x_k = 0 \}$ for the set of all money-distributions.

**Definition.** Let $\{B_k\}_{1 \leq k \leq n}$ be an $n$-partition of $\Omega$, $\pi \in \Pi$ and $x \in \mathcal{M}$. Define for all $1 \leq k \leq n$ and $i \in N$,
Here, \([a]_+\) denotes \(\max\{a, 0\}\). Furthermore, we define for all \(i \in N\),

\[
E_i(\pi, x) := \sum_{k=1}^n E^k_i(\pi, x).
\]

The map

\[
E : \Pi \times \mathbb{M} \to \mathbb{R}_+
\]

\[
(\pi, x) \mapsto \sum_{i \in N} E_i(\pi, x)
\]

is the envy-measure of the allocation \((B_{\pi(i)}, x_{\pi(i)})_{i \in N}\). \(\circ\)

**Remark.** Given an \(n\)-partition \(\{B_k\}_{1 \leq k \leq n}\) of \(\Omega\), the allocation \((B_{\pi(i)}, x_{\pi(i)})_{i \in N}\) is envy-free if and only if \(E(\pi, x) = 0\). \(\circ\)

The following proposition will turn out to be very useful in our analysis.

**Proposition 4.2.** Let \(\{B_k\}_{1 \leq k \leq n}\) be an \(n\)-partition of \(\Omega\) and let \(\varepsilon \geq 0\). Then for all \(\pi \in \Pi\) the set

\[
\mathcal{A}_\varepsilon(\varepsilon) := \{x \in \mathbb{M} | E(\pi, x) \leq \varepsilon\}
\]

is a (possible empty) compact set of \(\mathbb{R}^n\).

**Proof.** Let \(\varepsilon \geq 0\) and \(\{B_k\}_{1 \leq k \leq n}\) be an \(n\)-partition of \(\Omega\). It is easy to verify that the map

\[
x \mapsto E(\pi, x)
\]

is continuous on \(\mathbb{M}\) for every \(\pi \in \Pi\). Therefore, \(\mathcal{A}_\varepsilon(\varepsilon)\) is closed for every \(\pi \in \Pi\).

Let \(\pi \in \Pi\) and let \(x \in \mathcal{A}_\varepsilon(\varepsilon)\). Take \(i \in N\), then certainly \(E_i(\pi, x) \leq \varepsilon\). This means that for all \(1 \leq k \leq n\),

\[
V_i(B_k) + f_i(B_k)(x_k) - V_i(B_{\pi(i)}) - f_i(B_{\pi(i)})(x_{\pi(i)}) \leq \varepsilon.
\]

There exists at least one number \(1 \leq \ell \leq n\) such that \(x_\ell \geq 0\) (here \(\ell = \pi(i)\) is also one of the possibilities) and therefore, \(f_i(B_k)(x_\ell) \geq f_i(B_k)(0) = 0\). Hence,

\[
f_i(B_{\pi(i)})(x_{\pi(i)}) \geq V_i(B_\ell) + f_i(B_k)(x_\ell) - V_i(B_{\pi(i)}) - \varepsilon \\
\geq V_i(B_\ell) - V_i(B_{\pi(i)}) - \varepsilon \\
\geq \min_{1 \leq k \leq n} V_i(B_k) - V_i(B_{\pi(i)}) - \varepsilon.
\]

As the function \(f_i(B_{\pi(i)}): \mathbb{R} \to \mathbb{R}\) is strictly monotonic, this yields,

\[
x_{\pi(i)} \geq f_i^{1-1}(\min_{1 \leq k \leq n} V_i(B_k) - V_i(B_{\pi(i)}) - \varepsilon) =: m_{\pi(i)}.
\]

So, \(x_{\pi(i)} \geq m_{\pi(i)}\) for all \(i \in N\) whenever \(x \in \mathcal{A}_\pi(\varepsilon)\). Because \(\sum_{i \in N} x_{\pi(i)} = 0\), we also obtain that

\[
x_{\pi(i)} = - \sum_{j \neq i} x_{\pi(j)} \leq - \sum_{j \neq i} m_{\pi(j)} =: M_{\pi(i)}.
\]

Hence, \(m_{\pi(i)} \leq x_{\pi(i)} \leq M_{\pi(i)}\) for all \(i \in N\) whenever \(x \in \mathcal{A}_\pi(\varepsilon)\) and thus \(\mathcal{A}_\pi(\varepsilon)\) is a bounded set in \(\mathbb{R}^n\). \(\square\)
4.3 Envy-free allocations

Similarly, as in the proof of the existence of envy-free allocations by Alkan et al. (1991) in their setting, we start by investigating economies in which the functions \( f_{iB} : \mathbb{R} \rightarrow \mathbb{R} \) are \textit{kink functions} for all \( i \in N \) and \( B \subseteq \Omega \). To do so, we first repeat the definition of a kink function.

**Definition.** A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a \textit{kink function} if \( f \) is a strictly increasing continuous function on \( \mathbb{R} \) and for each point \( x \in \mathbb{R} \) there exists a number \( \delta > 0 \) and positive numbers \( \lambda, \mu \geq 0 \) such that

\[
    f(y) = \begin{cases} 
    f(x) + \lambda(y - x) & \text{for } x - \delta \leq y \leq x, \\
    f(x) + \mu(y - x) & \text{for } x \leq y \leq x + \delta.
    \end{cases}
\]

If \( \lambda \neq \mu \), then the point \( x \) is a \textit{kink point} of the kink function \( f \). We write \( \lambda = d^{-} f(x) \) and \( \mu = d^{+} f(x) \). We denote \( K \) as the family of all kink functions.

The following theorem, proved in Section 4.5, states that any \( n \)-partition of \( \Omega \) can be extended to an envy-free allocation whenever the preference relations can be represented by kink functions.

**Theorem 4.3K.** Let \( N, \Omega, (\preceq_i)_{i \in N} \) be an economy in which the preference relations are represented by utility functions \( U_i(C, x) = V_i(C) + f_{iC}(x) \) with \( f_{iC} \in K \) for all \( i \in N \) and \( C \subseteq \Omega \). If \( \{B_k\}_{1 \leq k \leq n} \) is an \( n \)-partition of \( \Omega \), then there exists a bijection \( \pi \in \Pi \) and a money-distribution \( x \in \mathbb{M} \) such that the allocation \( (B_{\pi(i)}, x_{\pi(i)})_{i \in N} \) is envy-free.

Using Theorem 4.3K and a basic result from calculus which states that every strictly increasing continuous function can be approximated uniformly by a kink function (see e.g., Courant and John (1974)), one can prove the existence of an envy-free allocation in an economy \( N, \Omega, (\preceq_i)_{i \in N} \) with \( \preceq_i \in \mathcal{R} \) for all \( i \in N \). That is, any \( n \)-partition of \( \Omega \) can be extended to an envy-free allocation.

**Theorem 4.3.** Let \( N, \Omega, (\preceq_i)_{i \in N} \) be an economy with \( \preceq_i \in \mathcal{R} \) for all \( i \in N \) and let \( \{B_k\}_{1 \leq k \leq n} \) be an \( n \)-partition of \( \Omega \). Then there exists a bijection \( \pi \in \Pi \) and a money-distribution \( x \in \mathbb{M} \) such that the allocation \( (B_{\pi(i)}, x_{\pi(i)})_{i \in N} \) is envy-free.

**Proof.** Let \( \{B_k\}_{1 \leq k \leq n} \) be an \( n \)-partition of \( \Omega \) and let the preference relations \( \preceq_i \in \mathcal{R} \) be represented by the utility functions \( U_i(C, x) = V_i(C) + f_{iC}(x) \) for all \( i \in N \).

Take \( \varepsilon > 0 \). Each of the functions \( f_{iC} \) can be approximated uniformly by kink functions, i.e., there are kink functions \( g_{iC} \in K \) such that

\[
    |g_{iC}(x) - f_{iC}(x)| < \frac{\varepsilon}{n^2} \quad \text{for all } x \in \mathbb{R}, \ i \in N \text{ and } C \subseteq \Omega.
\]

In the economy with the utility functions \( U'_i(C, x) := V_i(C) + g_{iC}(x) \) there exists according to Theorem 4.3K, a bijection \( \pi \in \Pi \) and a money-distribution \( x \in \mathbb{M} \) such that the allocation \( (B_{\pi(i)}, x_{\pi(i)})_{i \in N} \) is envy-free. Let \( i \in N \) and \( 1 \leq k \leq n \). Then

\[
    E^k_i(\pi, x) = \left[ U_i(B_k) + f_{iB_k}(x_k) - (V_i(B_{\pi(i)}) + f_{iB_{\pi(i)}}(x_{\pi(i)})) \right]_+
    \leq 2 \frac{\varepsilon}{n^2} + |V_i(B_k) + g_{iB_k}(x_k) - (V_i(B_{\pi(i)}) + g_{iB_{\pi(i)}}(x_{\pi(i)}))|_+
    = \frac{\varepsilon}{n^2} + |U'_i(B_k, x_k) - U'_i(B_{\pi(i)}, x_{\pi(i)})|_+.
\]
The last equality follows from the fact that the allocation \((B_{\pi(i)}, x_{\pi(i)})_{i \in N}\) is envy-free in the economy with utility functions \(U_i : 2^\Omega \times \mathbb{R} \rightarrow \mathbb{R}\). Hence, we obtain that
\[
E(\pi, x) = \sum_{i \in N} \sum_{k=1}^{n} E_i^k(\pi, x) \leq \varepsilon.
\]

So, for every \(\varepsilon > 0\) there exists a bijection \(\pi \in \Pi\) such that
\[
A_\pi(\varepsilon) = \{x \in M \mid E(\pi, x) \leq \varepsilon\} \neq \emptyset.
\]

Let \(\{\varepsilon_k\}_{k \in \mathbb{N}}\) be a sequence of positive numbers such that \(\varepsilon_k \downarrow 0\). Then for each \(k \in \mathbb{N}\) there is a bijection \(\pi_k \in \Pi\) such that \(A_{\pi_k}(\varepsilon_k) \neq \emptyset\). As \(\pi_k \in \Pi\) allows only finitely many possibilities, we may assume that for all \(k \in \mathbb{N}\), \(\pi_k = \pi\) for a certain \(\pi \in \Pi\). Furthermore, according to Proposition 4.2, the set \(A_{\pi}(\varepsilon_k)\) is compact for every \(k \in \mathbb{N}\). Hence, the sets \(\{A_{\pi}(\varepsilon_k)\}_{k \in \mathbb{N}}\) are compact, pairwise non-empty and moreover, the sequence is weakly decreasing in \(k\). Therefore,
\[
\bigcap_{k \in \mathbb{N}} A_{\pi}(\varepsilon_k) \neq \emptyset.
\]

For a point \(x\) in the intersection \(\bigcap_{k} A_{\pi}(\varepsilon_k)\) it holds that \(E(\pi, x) < \varepsilon_k\) for all \(k \in \mathbb{N}\). Hence, \(E(\pi, x) = 0\) and thus the allocation \((B_{\pi(i)}, x_{\pi(i)})_{i \in N}\) is envy-free.

So, in the economies with indivisible goods and money, studied in this chapter, there exists an allocation which is fair for all agents with respect to the normative concept of envy-freeness.

### 4.4 Envy-free and Pareto efficient allocations

This part of the chapter deals with the existence of a second concept for allocating the indivisible goods among the agents, that is, the concept of envy-freeness and Pareto efficiency. Let us start by repeating the definition of a Pareto efficient allocation.

**Definition.** Let \((B_{\pi(i)}, x_{\pi(i)})_{i \in N}\) be an allocation. Then the allocation \((C_{\phi(i)}, y_{\phi(i)})_{i \in N}\) is an **improvement** upon \((B_{\pi(i)}, x_{\pi(i)})_{i \in N}\) if
\[
U_i(C_{\phi(i)}, y_{\phi(i)}) > U_i(B_{\pi(i)}, x_{\pi(i)}) \quad \text{for all } i \in N.
\]

An allocation \((B_{\pi(i)}, x_{\pi(i)})_{i \in N}\) is **Pareto efficient** if it does not admit an improvement.

**Remark.** In most studies one makes a difference between weak Pareto efficiency and (strong) Pareto efficiency (see e.g., Chapter 3). However, since we consider preference relations which satisfy strict monotonicity and continuity in money and moreover the monetary compensations may be negative, these two concepts are the same. Indeed, given a weak improvement upon an allocation (i.e., the strict inequalities may be tight except for at least one agent), it is a straightforward exercise, by using strict monotonicity and continuity, to construct a (strong) improvement.
4.4 Envy-free and Pareto efficient allocations

4.4.1 Envy-free and Pareto efficient allocations in general

In this subsection we give an example of an economy \( \langle N; \Omega, (\preceq_i)_{i \in N} \rangle \), with \( \preceq_i \in \mathcal{R} \) for all \( i \in N \), in which the set of envy-free allocations and the set of Pareto efficient allocations are both non-empty but disjoint.

**Example.** Let \( \langle N; \Omega, (\preceq_i)_{i \in N} \rangle \) be an economy with \( N := \{1, 2\} \) and \( \Omega := \{\alpha, \beta\} \). The utility functions \( U_i : 2^\Omega \times \mathbb{R} \rightarrow \mathbb{R} \) for \( i = 1, 2 \) are given by:

<table>
<thead>
<tr>
<th>( \emptyset, x )</th>
<th>( {\alpha}, x )</th>
<th>( {\beta}, x )</th>
<th>( {\alpha, \beta}, x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>agent 1</td>
<td>10 + x</td>
<td>11 + 2 \cdot x</td>
<td>12 + 2 \cdot x</td>
</tr>
<tr>
<td>agent 2</td>
<td>x</td>
<td>1 \frac{19}{20} \cdot x</td>
<td>19 + 1 \frac{19}{20} \cdot x</td>
</tr>
</tbody>
</table>

The following allocations are possible:

(a) Assume that the allocation \( ((\{\alpha, \beta\}, -x), (\emptyset, x)) \) is envy-free for a certain \( x \in \mathbb{R} \). This means that \( U_1((\{\alpha, \beta\}, -x) = 12 - 2 \cdot x \geq x = U_1(\emptyset, x) \) and \( U_2(\emptyset, x) = x \geq 19 - 1 \frac{19}{20} \cdot x = U_2((\{\alpha, \beta\}, -x)) \). This yields, \( x \leq 4 \) and \( x \geq 6 \frac{26}{27} \). So, the allocation \( ((\{\alpha, \beta\}, -x), (\emptyset, x)) \) is never envy-free.

The possible utility levels of the allocation \( ((\{\alpha, \beta\}, -x), (\emptyset, x)) \) fill the line \([\alpha \beta \emptyset] \) with equation \( u_1 + 2 \cdot u_2 = 12 \) (see Figure 6).

(b) Assume that the allocation \( ((\{\alpha\}, -x), (\{\beta\}, x)) \) is envy-free for a certain \( x \in \mathbb{R} \). This means that \( 10 - x \geq 11 + 2 \cdot x \) and \( 1 \frac{19}{20} \cdot x \geq -x \). This yields, \( x \geq 0 \) and \( x \leq -\frac{1}{2} \). So, the allocation \( ((\{\alpha\}, -x), (\{\beta\}, x)) \) is not envy-free for any \( x \in \mathbb{R} \).

The possible utility levels of the allocation \( ((\{\alpha\}, -x), (\{\beta\}, x)) \) fill the line \([\alpha \beta \emptyset] \) with equation \( 1 \frac{26}{27} \cdot u_1 + u_2 = 19 \frac{1}{2} \).

(c) Assume that the allocation \( ((\{\beta\}, x), (\{\alpha\}, -x)) \) is envy-free for a certain \( x \in \mathbb{R} \). Then \( 11 + 2 \cdot x \geq 10 - x \) and \( -x \geq 1 \frac{19}{20} \cdot x \) which yields, \( -\frac{1}{3} \leq x \leq 0 \). Thus, the allocation \( ((\{\beta\}, x), (\{\alpha\}, -x)) \) is envy-free whenever \( -\frac{1}{3} \leq x \leq 0 \).

The possible utility levels of the allocation \( ((\{\beta\}, x), (\{\alpha\}, -x)) \) fill the line \([\beta \alpha \emptyset] \) with equation \( u_1 + 2 \cdot u_2 = 11 \).

(d) Assume that the allocation \( ((\emptyset, x), (\{\alpha, \beta\}, -x)) \) is envy-free for a certain \( x \in \mathbb{R} \). Then \( x \geq 12 - 2 \cdot x \) and \( 19 - 1 \frac{19}{20} \cdot x \geq x \) which yields, \( 4 \leq x \leq 6 \frac{26}{27} \). Thus, the allocation \( ((\emptyset, x), (\{\alpha, \beta\}, -x)) \) is envy-free whenever \( 4 \leq x \leq 6 \frac{26}{27} \).

The possible utility levels of the allocation \( ((\emptyset, x), (\{\alpha, \beta\}, -x)) \) fill the line \([\emptyset \alpha \beta \emptyset] \) with equation \( 1 \frac{26}{27} \cdot u_1 + u_2 = 19 \).
The envy-free allocations correspond with the dashed part of the line \([\alpha, \beta]\) and with the dashed part of the line \([\emptyset, \alpha \beta]\) (Case (c) and Case (d), respectively).

The Pareto efficient allocations correspond with the union of the dark part of the line \([\alpha \beta, \emptyset]\) and of the dark part of the line \([\alpha, \beta]\) (Case (a) and Case (b), respectively).

Remark. If we define in the previous example for each agent \(i \in N\) a new utility function \(U'_i : 2^\Omega \times \mathbb{R} \rightarrow \mathbb{R}\) as,

\[
U'_i(B, x) := \max_{C \subseteq B} U_i(C, x)
\]

for all \(B \subseteq \Omega\) and \(x \in \mathbb{R}\),

then all the indivisible goods have a non-negative marginal value for both agents, i.e., all the indivisible goods are desired by both agents. It is left to the reader to check that in this case the sets of envy-free allocations and Pareto efficient allocations are also disjoint.
4.4 Envy-free and Pareto efficient allocations

4.4.2 Sufficient conditions for compatibility

The example given in Subsection 4.4.1 illustrates that in general the concept of envy-freeness and the concept of Pareto efficiency are incompatible. However, this subsection gives two conditions for economies, each sufficient for the compatibility, namely, the existence of a price equilibrium and the existence of a weakly dominant n-partition. Finally in this subsection, we look at economies in which the utilities are quasi-linear. But first we give the definition of a price equilibrium.

**Definition.** An allocation \((B_p, x_p)\) is a price equilibrium, if there exists a price vector \(p \in \mathbb{R}^\Omega\) with the following two properties:

1. \(p(B_p) + x_p = \frac{1}{n} p(\Omega)\) for all \(i \in N\), (budget constraints)
2. If \(U_i(C, y) > U_i(B_p, x_p)\) for a certain \(C \subseteq \Omega\), \(y \in \mathbb{R}\) and \(i \in N\), then \(p(C) + y > \frac{1}{n} p(\Omega)\), (maximality conditions)

For all \(C \subseteq \Omega\), we denote \(p(C) := \sum_{\alpha \in C} p_\alpha\).

In contrast to the definition given in Section 3.4, in this definition of a price equilibrium the budget of an agent is defined as ‘one \(n\)-th of the price of all the indivisible goods added together’. The reason for this alteration is the absence of initial endowments. However, the interpretation of the budget constraints as well as the maximality conditions remain the same as given in Section 3.4.

The existence of a price equilibrium is sufficient for the existence of an envy-free and Pareto efficient allocation.

**Proposition 4.4.** If \(\langle N, \Omega, (\preceq_i)_{i \in N} \rangle\) is an economy with \(\preceq_i \in \mathcal{R}\) for all \(i \in N\) in which there exists a price equilibrium, then there exists an envy-free and Pareto efficient allocation.

**Proof.** Let \((B_p, x_p)_{i \in N}\) be a price equilibrium supported by the price vector \(p \in \mathbb{R}^\Omega\).

Suppose \(U_i(B_p, x_p) < U_i(B_p, x_p)\) for some \(i, j \in N\). Then, according to the maximality conditions, it follows that \(p(B_p) + x_p > \frac{1}{n} p(\Omega)\). Contradiction with the budget constraints. So, the allocation is envy-free.

Suppose the allocation \((C_{\varphi}, y_{\varphi})\) is an improvement upon \((B_p, x_p)\). This means that \(U_i(C_{\varphi}, y_{\varphi}) > U_i(B_p, x_p)\) for all \(i \in N\) and therefore, according to the maximality conditions,

\[p(C_{\varphi}) + y_{\varphi} > \frac{1}{n} p(\Omega)\]

for all \(i \in N\).

Taking the sum over all \(i \in N\) and using the fact that \(\sum_{i \in N} y_{\varphi(i)} = 0\) yields a contradiction. \(\square\)

So, the existence of a price equilibrium assures the existence of an envy-free and Pareto efficient allocation. However, there may be envy-free and Pareto efficient allocations which are not supported by a price vector, even in an economy which has price equilibria. We illustrate this phenomenon at the end of the section by means of an example.
Remark. In the example given in Subsection 4.4.1, there exists no weakly dominant partition. This can be best seen in Figure 6. In this figure the Pareto efficient utility levels are obtained via two different $n$-partitions (namely, via the 2-partitions $\{\{\alpha\},\{\beta\}\}$ and $\{\{\alpha,\beta\},\emptyset\}$). Hence, the assumption of a weakly dominant $n$-partition cannot simply be omitted for the compatibility of an envy-freeness and Pareto efficiency.

Finally in this section, economies are studied in which for all agents the marginal utility of money is constant. That is, we assume that for each agent $i \in N$ the preference relation $\preceq_i \in \mathcal{R}$ to satisfy the following additional property:
4.4 Envy-free and Pareto efficient allocations

for every \( B \subseteq \Omega \) and every \( x \in \mathbb{R} \) such that \((B, 0) \sim_{i} (\emptyset, x)\) it holds that 
\((B, d) \sim_{i} (\emptyset, x + d)\) for all \( d \in \mathbb{R}\).

Using this property it is easy to verify that \( f_i(B)(x) = f_i(B)(0 + x) = f_i(B)(0) + x = x \) for all \( B \subseteq \Omega \) and \( i \in N \) and therefore, \( U_i(B, x) = V_i(B) + x \) for all \( B \subseteq \Omega \), \( x \in \mathbb{R} \) and \( i \in N \) (see also Proposition 3.1). Thus, the utility functions are quasi-linear. In this quasi-linear setting we prove the existence of an envy-free and Pareto efficient allocation. The following well-known proposition states a necessary and sufficient condition for an allocation to be Pareto efficient allocations are exactly those allocations which maximize social welfare. Note that in Proposition 4.6 there are no conditions on the money-distribution. The reason is exactly that there is an abundance of money (therefore, Proposition 4.6 is no longer true for the exchange economies in Chapter 3).

Next, we prove that the \( n \)-partition used in such an allocation is in fact weakly dominant.

**Proposition 4.6.** If \( \langle N, \Omega, (\preceq_{i})_{i \in N} \rangle \) is an economy with quasi-linear utilities, then an allocation \( (B_{\pi(i)}, x_{\pi(i)})_{i \in N} \) is Pareto efficient if and only if

\[
\sum_{i \in N} V_i(B_{\pi(i)}) \geq \sum_{i \in N} V_i(C_{\varphi(i)})
\]

for all \( n \)-partitions \( \{C_k\}_{1 \leq k \leq n} \) and all bijections \( \varphi \in \Pi \).

**Proof.** \( \Rightarrow \) Let \( (B_{\pi(i)}, x_{\pi(i)})_{i \in N} \) be a Pareto efficient allocation. Suppose there exists an \( n \)-distribution \( \{C_k\}_{1 \leq k \leq n} \) such that \( \sum_{i \in N} V_i(B_{\pi(i)}) < \sum_{i \in N} V_i(C_{\varphi(i)}) \). Define

\[
\delta := \frac{1}{n} \sum_{i \in N} \left[ V_i(C_{\pi(i)}) - V_i(B_{\pi(i)}) \right],
\]

\[
y_{\pi(i)} := V_i(B_{\pi(i)}) - V_i(C_{\pi(i)}) + x_{\pi(i)} + \delta \quad \text{(for all } i \in N)\).
\]

Then \( \delta > 0 \) and thus \( U_i(C_{\pi(i)}, y_{\pi(i)}) > U_i(B_{\pi(i)}, x_{\pi(i)}) \) for all \( i \in N \). Contradiction.

\( \Leftarrow \) Let \( (B_{\pi(i)}, x_{\pi(i)})_{i \in N} \) be an allocation with \( \sum_{i \in N} V_i(B_{\pi(i)}) \geq \sum_{i \in N} V_i(C_{\varphi(i)}) \) for all \( n \)-partitions \( \{C_k\}_{1 \leq k \leq n} \) and all bijections \( \varphi \in \Pi \). Suppose this allocation admits an improvement. Then there exists an allocation \( (C_{\pi(i)}, y_{\pi(i)})_{i \in N} \) such that

\[
V_i(C_{\pi(i)}) + y_{\pi(i)} > V_i(B_{\pi(i)}) + x_{\pi(i)} \quad \text{for all } i \in N.
\]

Hence, \( \sum_{i \in N} V_i(C_{\pi(i)}) > \sum_{i \in N} V_i(B_{\pi(i)}) \). Contradiction. \( \square \)

Proposition 4.6 in fact states that in an economy in which the utility functions are quasi-linear (and in which there is an abundance of money), the Pareto efficient allocations are exactly those allocations which maximize social welfare. Note that in Proposition 4.6 there are no conditions on the money-distribution. The reason is exactly that there is an abundance of money (therefore, Proposition 4.6 is no longer true for the exchange economies in Chapter 3).

Next, we prove that the \( n \)-partition used in such an allocation is in fact weakly dominant.

**Proposition 4.7.** If \( \langle N, \Omega, (\preceq_{i})_{i \in N} \rangle \) is an economy with quasi-linear utilities, then there exists a weakly dominant \( n \)-partition.

**Proof.** Let \( \{B_k\}_{1 \leq k \leq n} \) be an \( n \)-partition and \( \pi \in \Pi \) be a bijection such that
The following result can be immediately derived from Proposition 4.5 and Proposition 4.7. Hence, the allocation \( (C_{\varphi(i)}, y_{\varphi(i)})_{i \in N} \) is envy-free and Pareto efficient.

\[
\sum_{i \in N} V_i(B_{\pi(i)}) \geq \sum_{i \in N} V_i(C_{\varphi(i)})
\]

for all \( n \)-partitions \( \{C_k\}_{1 \leq k \leq n} \) and all bijections \( \varphi \in \Pi \).

Let \( (C_{\varphi(i)}, y_{\varphi(i)})_{i \in N} \) be any allocation. Define, for all \( i \in N \),

\[
x_{\pi(i)} := V_i(C_{\varphi(i)}) - V_i(B_{\pi(i)}) + y_{\varphi(i)} + \frac{1}{n} \sum_{i \in N} [V_i(B_{\pi(i)}) - V_i(C_{\varphi(i)})].
\]

Then \( (B_{\pi(i)}, x_{\pi(i)})_{i \in N} \) is an allocation. Furthermore, for all \( i \in N \),

\[
U_i(B_{\pi(i)}, x_{\pi(i)}) = V_i(B_{\pi(i)}) + x_{\pi(i)} \geq V_i(C_{\varphi(i)}) + y_{\varphi(i)} = U_i(C_{\varphi(i)}, y_{\varphi(i)}).
\]

Hence, the \( n \)-partition \( \{B_k\}_{1 \leq k \leq n} \) is weakly dominant. \( \square \)

The following result can be immediately derived from Proposition 4.5 and Proposition 4.7.

**Corollary 4.8.** If \( \langle N, \Omega, (\leq_i)_{i \in N} \rangle \) is an economy with \( \leq_i \in \mathcal{R} \) and the marginal utility of money is constant for all \( i \in N \), so the utility functions are quasi-linear, then there exists an envy-free and Pareto efficient allocation.

\( \square \)

Although, according to Corollary 4.8 the existence of an envy-free and Pareto efficient allocation is guaranteed in an economy with quasi-linear utilities, there may remain Pareto efficient allocations that are not envy-free and envy-free allocations that are not Pareto efficient. The latter is in contrast with Theorem 1 in Alkan et al. (1991) in their setting. This is shown in the following example. The example is also used to illustrate that an envy-free and Pareto efficient allocation is not necessarily a price equilibrium, even in an economy with a price equilibrium.

**Example.** Let \( \langle N, \Omega, (\leq_i)_{i \in N} \rangle \) be an economy with \( N := \{1, 2\} \) and \( \Omega := \{\alpha, \beta\} \). The utility functions \( U_i : 2^\Omega \times \mathbb{R} \rightarrow \mathbb{R} \) for \( i = 1, 2 \) are quasi-linear and given by:

<table>
<thead>
<tr>
<th></th>
<th>( \emptyset, x )</th>
<th>( {\alpha}, x )</th>
<th>( {\beta}, x )</th>
<th>( {\alpha, \beta}, x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>agent 1</td>
<td>( x )</td>
<td>( 8 + x )</td>
<td>( 9 + x )</td>
<td>( 13 + x )</td>
</tr>
<tr>
<td>agent 2</td>
<td>( x )</td>
<td>( 5 + x )</td>
<td>( 7\frac{1}{2} + x )</td>
<td>( 14 + x )</td>
</tr>
</tbody>
</table>

The allocation \( (\emptyset, 7), (\{\alpha, \beta\}, -7) \) is envy-free. However, it is not Pareto efficient. Indeed, \( V_1(\emptyset) + V_2(\{\alpha, \beta\}) = 14 < 15\frac{1}{2} = V_1(\{\alpha\}) + V_2(\{\beta\}) \) and therefore, according to Proposition 4.6, it cannot be Pareto efficient.

The allocation \( (\{\alpha\}, 2), (\{\beta\}, -2) \) is Pareto efficient. However, it is not envy-free, since \( U_2(\{\beta\}, -2) = 5\frac{1}{2} < 7 = U_2(\{\alpha\}, 2) \).

The allocation \( (\{\alpha\}, 1), (\{\beta\}, -1) \) is envy-free and Pareto efficient. However, it is not a price equilibrium as we now demonstrate. Suppose \( (\{\alpha\}, 1), (\{\beta\}, -1) \) is a price equilibrium supported by a price vector \( p := (p_\alpha, p_\beta) \in \mathbb{R}^\Omega \). Because

\[
U_2(\emptyset, 7) = 7 > 6\frac{1}{2} = U_2(\{\beta\}, -1) \quad \text{and} \quad U_2(\{\alpha, \beta\}, -7) = 7 > 6\frac{1}{2} = U_2(\{\beta\}, -1),
\]

The following example is also used to illustrate that an envy-free and Pareto efficient allocation is not necessarily a price equilibrium, even in an economy with a price equilibrium.
we have according to the maximality conditions that
\[ 7 > \frac{1}{2}(p_\alpha + p_\beta) \quad \text{and} \quad p_\alpha + p_\beta - 7 > \frac{1}{2}(p_\alpha + p_\beta). \]
This yields a contradiction. However, it is a straightforward calculation to illustrate that the allocation \(((\{a\}, \frac{1}{2}), (\{b\}, -\frac{1}{2}))\), supported by the price vector \(p := (6\frac{1}{2}, 7\frac{1}{2})\), is a price equilibrium and therefore, according to Proposition 4.4, also envy-free and Pareto efficient.\(\star\)

4.5 Proof of Theorem 4.3\(\star\)

This final section of the chapter contains a proof for Theorem 4.3. Throughout this section economies \(\langle N, \Omega, (\succeq_i)_{i \in N}\rangle\) are considered in which the given preference relations \(\succeq_i \in \mathcal{R}\) for all \(i \in N\) are represented by kink functions, i.e.,
\[ U_i(B, x) = V_i(B) + f_{iB}(x) \quad \text{with} \quad f_{iB} \in \mathcal{K} \quad \text{for all} \quad B \subseteq \Omega \quad \text{and} \quad x \in \mathbb{R}. \]
Recall that if \(f \in \mathcal{K}\) is a kink function we denote for every \(x \in \mathbb{R}\),
\[ d^+ f(x) := \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x} \quad \text{and} \quad d^- f(x) := \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x}. \]
Note that \(d^+ f(x) = d^- f(x) = \frac{df}{dx}(x)\) almost everywhere, since a kink function has a countable number of kink points.
Before we prove Theorem 4.3\(\star\), we first recall a result from combinatorial optimization (see e.g., Nemhauser and Wolsey (1988)).

**Lemma 4.9.** Let \(I\) and \(J\) be two finite index sets with \(|I| = |J|\). Furthermore, let \(D_{ij} \in \mathbb{R} \cup \{-\infty, \infty\}\) for all \(i \in I\) and \(j \in J\), such that for all \(i \in I\), \(D_{\varphi(i)} \in \mathbb{R}\) for some bijection \(\varphi : I \rightarrow J\).

(i) If \(D_{ij} < \infty\) for all \(i \in I\) and \(j \in J\), then there are real numbers \(\{v_j\}_{j \in J}\) and a bijection \(\pi : I \rightarrow J\) such that
\[ D_{\pi(i)} - v_{\pi(j)} \leq D_{ij} \quad \text{for all} \quad i, k \in I \quad \text{and} \quad D_{\pi(i)} \neq -\infty \quad \text{for all} \quad i \in I, \]

(ii) If \(D_{ij} > -\infty\) for all \(i \in I\) and \(j \in J\), then there are real numbers \(\{v_j\}_{j \in J}\) and a bijection \(\pi : I \rightarrow J\) such that
\[ D_{\pi(i)} - v_{\pi(i)} \geq D_{ij} \quad \text{for all} \quad i, k \in I \quad \text{and} \quad D_{\pi(i)} \neq \infty \quad \text{for all} \quad i \in I. \]

**Proof.** (i) Let \(D_{ij} \in \mathbb{R} \cup \{-\infty\}\) for all \(i \in I\) and \(j \in J\). Define the linear program (LP)
\[
\begin{align*}
\text{maximize} & \quad \sum_{i \in I} \sum_{j \in J} D_{ij} \cdot X_{ij} \\
\text{subject to:} & \quad X_{ij} \geq 0 \quad \text{for all} \quad i \in I \quad \text{and} \quad j \in J, \\
& \quad \sum_{i \in I} X_{ij} = \sum_{j \in J} X_{ij} = 1 \quad \text{for all} \quad j \in J \quad \text{and} \quad i \in I, \quad \text{respectively.}
\end{align*}
\]
Since there exists a bijection $\varphi : I \rightarrow J$ such that $D_{i\varphi(i)} \in \mathbb{R}$ for all $i \in I$ and because the feasible region of (LP) is compact it follows that (LP) has an optimal solution. Moreover, the matrix $(X_{ij})_{i \in I, j \in J}$ is double-stochastic. Hence, there exists a bijection $\pi : I \rightarrow J$ such that for all $i \in I$,

$$X_{\pi(i)} := 1 \text{ and } X_{\pi(k)} := 0 \text{ whenever } k \neq i$$

is an optimal solution of (LP). Furthermore, $D_{\pi(i)} \in \mathbb{R}$ for all $i \in I$.

By complementary slackness if follows that the dual program (LP*) has an optimal solution

$$\{u_i, v_{\pi(k)}\}_{i, k \in I} \text{ such that } u_i + v_{\pi(i)} = D_{\pi(i)} \text{ and } u_i + v_{\pi(k)} \geq D_{\pi(k)} \text{ for all } i, k \in I.$$  

So, we can conclude that there are real numbers $\{v_j\}_{j \in J}$ and a bijection $\pi : I \rightarrow J$ such that $D_{\pi(i)} - v_{\pi(k)} \leq D_{\pi(i)} - v_{\pi(i)}$ for all $i, k \in I$ and $D_{\pi(i)} \neq -\infty$ for all $i \in I$.

(ii) If $D_{ij} \in \mathbb{R} \cup \{\infty\}$ for all $i \in I$ and $j \in J$ and we minimize (in stead of maximize) in the (LP) above, a similar reasoning can be used to prove the second statement. The details are left to the reader.  

The proof of Theorem 4.3 K boils down to the following proposition. In the proof of this proposition Lemma 4.9 is used.

**Proposition 4.10.** Let $\{B_k\}_{1 \leq k \leq n}$ be an $n$-partition. If $\varphi \in \Pi$ is a bijection and $y \in \mathbb{M}$ a money-distribution such that $E(\varphi, y) > 0$, then there exists a bijection $\pi \in \Pi$ and a money-distribution $x \in \mathbb{M}$ such that $E(\pi, x) < E(\varphi, y)$.

**Proof.** Let $\{B_k\}_{1 \leq k \leq n}$ be an $n$-partition of $\Omega$. Let $\varphi \in \Pi$ and $y \in \mathbb{M}$ such that $E(\varphi, y) > 0$. We construct a new bijection $\pi \in \Pi$ and a new money-distribution $x \in \mathbb{M}$ such that $E(\pi, x) < E(\varphi, y)$.

As a tool to keep track of envy, we introduce a so-called *envy-graph* belonging to the partition $\{B_k\}_{1 \leq k \leq n}$, bijection $\varphi \in \Pi$ and money-distribution $y \in \mathbb{M}$. The same method is used in Klijn (2000) in the case that all utility functions are quasi-linear. The nodes of the graph are identified with the bundles $(B_k, y_k)$. There is a weak arc from $(B_k, y_k)$ to $(B_\ell, y_\ell)$ (with $\ell \neq k$) if

$$U_i(B_k, y_k) = U_i(B_\ell, y_\ell) \text{ and } k = \varphi(i).$$

**Figure 7(i).** A weak arc.

There is a strong arc from $(B_k, y_k)$ to $(B_\ell, y_\ell)$ (with $\ell \neq k$) if

$$U_i(B_k, y_k) < U_i(B_\ell, y_\ell) \text{ and } k = \varphi(i).$$

**Figure 7(ii).** A strong arc.

Agent $i \in N$ is the owner of node $(B_k, y_k)$ (or shortly, node $B_k$) if $\varphi(i) = k$, i.e., the bundle $(B_k, y_k)$ is assigned, by the bijection $\varphi \in \Pi$, to agent $i$. So, if agent $i$ is indifferent between
the bundles \((B_k, y_k)\) and \((B_\ell, y_\ell)\), there is a weak arc from node \(B_k\) to node \(B_\ell\). If he strictly prefers the bundle \((B_\ell, y_\ell)\) above his own bundle \((B_k, y_k)\), there is a strong arc from node \(B_k\) to node \(B_\ell\). If he dislikes the bundle \((B_\ell, y_\ell)\), there is no arc. Here we emphasize that if \(\varphi(i) = k\) and \(\varphi(j) = \ell\), then a strong/weak/no arc from node \(B_k\) to node \(B_\ell\) has a different meaning that a strong/weak/no arc from node \(B_\ell\) to node \(B_k\). In the first case the arc follows from agent \(i\)'s point of view (and his utility function \(U_i\)), while the second case is in the perspective of agent \(j\) and his utility function \(U_j\). Also note that an allocation is envy-free if and only if the envy-graph does not contain a strong arc.

If the envy-graph of the allocation \((B_{\varphi(i)}, y_{\varphi(i)})\) in \(N\) contains a strong arc, then to improve the situation, i.e., decrease the envy \(E(\varphi, y)\), agents are allowed to move to other node (i.e., the bijection \(\varphi \in \Pi\) changes) and/or the amounts of money \(y_k\) can change (i.e., the money-distribution \(y \in \mathcal{M}\) changes).

Assume that the envy-graph of \((\varphi, y)\) has a circuit containing a strong arc. Then it is easy to construct a bijection \(\pi \in \Pi\) and money-distribution \(x \in \mathcal{M}\) such that \(E(\pi, x) < E(\varphi, y)\). The bundles \((B_k, y_k)\) remain the same (i.e., \(x := y\)) but the agents owning a bundle in the circuit move one step in the direction of the circuit. Then every agent obtains a weakly better bundle and at least one agent obtains a strictly better bundle. The value of \(E_i(\pi, y)\) is at most \(E_i(\varphi, y)\) for all \(i \in N\) and \(E_i(\pi, y)\) is strictly less than \(E_i(\varphi, y)\), if the strong arc in the circuit connects \(k^* := \varphi(i^*)\) to \(\ell^* := \pi(i^*)\).

\[
\begin{align*}
\varphi: & \quad (B_k, y_k) \quad \rightarrow \quad (B_\ell, y_\ell) \\
\pi: & \quad (B_\ell, y_\ell) \quad \rightarrow \quad (B_k, y_k)
\end{align*}
\]

Figure 8(i). A circuit containing a strong arc. The strong arc disappears if the agents move one step in the direction of the circuit, i.e., the bijection \(\varphi\) is changed into bijection \(\pi\).

If the envy-graph of \((\varphi, y)\) does not have a circuit containing a strong arc then it is more difficult to construct \(\pi \in \Pi\) and \(x \in \mathcal{M}\) such that \(E(\pi, x) < E(\varphi, y)\). Consider an arbitrarily chosen strong arc in the graph from, say from node \(B_{k^*}\) to node \(B_{\ell^*}\). The nodes in the graph are labeled as follows. Node \(B_{k^*}\) gets label \(\oplus\), node \(B_{\ell^*}\) gets label \(\ominus\). A node \(B_k\) \((k \neq k^*, \ell^*)\) gets label \(\oplus\) if there is a path of weak or strong arcs from \(B_k\) to \(B_{k^*}\). A node \(B_\ell\) \((\ell \neq k^*, \ell^*)\) gets label \(\ominus\) if there is a path of weak or strong arcs arcs from \(B_{\ell^*}\) to \(B_\ell\). Because there is no cycle with a strong arc, it is not possible that a node gets both labels \(\oplus\) and \(\ominus\). The remaining nodes (that have neither label \(\oplus\) nor label \(\ominus\)) get label \(\circ\). We define the following partition of \(N\):

\[
\begin{align*}
N^{\oplus} & := \{ i \in N \mid \text{node } B_{\varphi(i)} \text{ has label } \oplus \}, \\
N^{\ominus} & := \{ i \in N \mid \text{node } B_{\varphi(i)} \text{ has label } \ominus \}, \\
N^{\circ} & := \{ i \in N \mid \text{node } B_{\varphi(i)} \text{ has label } \circ \}.
\end{align*}
\]
Indeed, as far as there are no circuits with strong arcs, the equality $D_{H}$ holds. Hence, there are real numbers $m\in\mathbb{R}$ for the nodes with label $\ominus$ and the continuity in money, these two numbers can be chosen in such a way that no new strong arcs arise from a node labeled $\ominus$ to a node labeled $\oplus$, and that no new strong arc arises from a node labeled $\ominus$ to a node labeled $\ominus$. We refer to Klijn (2000) for all the details.

However, agents have a different valuation for money. Nevertheless, since the utility functions are kink functions, this primary idea, as described above, can be modified in such a way that no new strong arcs arise between the nodes. Next to increasing the amounts of money in the nodes with label $\oplus$ and by decreasing the amounts of money in the nodes with label $\ominus$. For the nodes with label $\ominus$ nothing changes. If agents would have the same valuation for money (for example, if they have quasi-linear utility functions), then the amount of money in each node labeled $\oplus$ can be increased with the same number. For each node labeled $\ominus$ money can be decreased with a fixed number. By doing so, no new strong arc will arise between nodes with label $\ominus$ or label $\ominus$, respectively. Furthermore, by the definitions of the labels and the continuity in money, these two numbers can be chosen in such a way that no new strong arc arises from a node labeled $\ominus$ to a node labeled $\oplus$ or $\ominus$, and that no new strong arc arises from a node labeled $\ominus$ to a node labeled $\ominus$. We refer to Klijn (2000) for all the details.

First, we look at the agents in $N_\oplus$. Define for all $i, j \in N_\oplus$

$$D_{\ominus \varphi(j)}(i) := \begin{cases} \log(d^{+}f_{i}B_{\varphi(j)}(y_{\varphi(i)})) & \text{if } U_{i}(B_{\varphi(i)}, y_{\varphi(i)}) \leq U_{i}(B_{\varphi(j)}, y_{\varphi(j)}), \\ -\infty & \text{if } U_{i}(B_{\varphi(i)}, y_{\varphi(i)}) > U_{i}(B_{\varphi(j)}, y_{\varphi(j)}). \end{cases}$$

Note that $D_{\ominus \varphi(j)}(i) \in \mathbb{R}$ for all $i \in N_\oplus$, since $d^{+}f_{i}B_{\varphi(j)}(y_{\varphi(i)}) > 0$ for all $i \in N_\oplus$. So, we can apply Lemma 4.9(i) with $I = N_\oplus$, $J = \varphi(N_\ominus)$ and $D_{ik} = D_{\ominus \varphi(j)}(i)$ for all $i, j \in N_\oplus$.

Hence, there are real numbers $\{v_{i}\}_{i \in \varphi(N_\ominus)}$ and a bijection $\pi_\ominus : N_\ominus \rightarrow \varphi(N_\ominus)$ such that $D_{\ominus \pi_\ominus(i)}(i) \in \mathbb{R}$ for all $i \in N_\ominus$ and

$$D_{\ominus \pi_\ominus(j)}(i) - v_{\pi_\ominus(j)} \leq D_{\ominus \pi_\ominus(i)} - v_{\pi_\ominus(i)} \text{ for all } i, j \in N_\ominus. \tag{1}$$

As $D_{\ominus \pi_\ominus(i)}(i) \in \mathbb{R}$ for all $i \in N_\ominus$ we also obtain, by definition of $D_{\ominus}$ and the fact that there are no circuits with strong arcs, the equality

$$U_{i}(B_{\varphi(i)}, y_{\varphi(i)}) = U_{i}(B_{\pi_\ominus(i)}, y_{\pi_\ominus(i)}) \text{ for all } i \in N_\ominus. \tag{2}$$

Indeed, as far as $\varphi(i) \neq \pi_\ominus(i)$ for certain $i \in N_\ominus$, there is a weak/strong arc from node $(B_{\varphi(i)}, y_{\varphi(i)})$ to node $(B_{\pi_\ominus(i)}, y_{\pi_\ominus(i)})$. This follows from the definition of $D_{\ominus}$ and the fact

---

**Figure 8(ii).** The labels in the envy-graph if there are no circuits containing a strong arc, but there is a strong arc. Here, this arc is depicted in gray and from node $(B_{k}, y_{k})$ (labeled $\oplus$) to node $(B_{r}, y_{r})$ (labeled $\ominus$).
4.5 Proof of Theorem 4.3K

that $D^{\oplus}_{\pi^{\oplus}(i)} \in \mathbb{R}$ for all $i \in N^{\oplus}$. But, each of these arcs are in a circuit and therefore, they are all weak arcs (since there is no circuit containing a strong arc). Hence, $U_i(B_{\phi(i)}, y_{\phi(i)}) = U_i(B_{\pi^{\oplus}(i)}, y_{\pi^{\oplus}(i)})$ for all $i \in N^{\oplus}$.

The bijection $\pi^{\oplus} : N^{\oplus} \rightarrow \varphi(N^{\oplus})$ is used to allocate the bundles with label $\oplus$ among the agents in $N^{\oplus}$. Furthermore, the numbers $\{v_j\}_{j \in \varphi(N^{\oplus})}$ are used to increase the amounts of money in the bundles with label $\ominus$. To be more precise, if for all $i \in N^{\oplus}$ we denote

$$a_{\pi^{\oplus}(i)} := e^{+y_{\pi^{\oplus}(i)}}$$

then agent $i \in N^{\oplus}$ receives the bundle

$$(B_{\pi^{\oplus}(i)}, x_{\pi^{\oplus}(i)}) \text{ with } x_{\pi^{\oplus}(i)} := y_{\pi^{\oplus}(i)} + \varepsilon \cdot a_{\pi^{\oplus}(i)}.$$  

The number $\varepsilon > 0$ is yet to be determined. For the moment, we select $\varepsilon > 0$ such that:

(P1) For all $i \in N$ and every $1 \leq k \leq n$ such that node $B_k$ has label $\oplus$, the function $f_iB_k$ is linear on the interval $[y_k, y_k + \varepsilon \cdot a_k]$.

(P2) For all $i \in N^{\ominus}$ and every $1 \leq k \leq n$ such that node $B_k$ has label $\oplus$, $\varepsilon > 0$ satisfies

$$U_i(B_k, y_k + \varepsilon \cdot a_k) \leq U_i(B_{\phi(i)}, y_{\phi(i)}),$$

(P3) For all $i \in N^{\ominus}$ and every $1 \leq k \leq n$ such that node $B_k$ has label $\ominus$, $\varepsilon > 0$ satisfies

$$U_i(B_{\pi^{\ominus}(i)}, y_{\pi^{\ominus}(i)}) = U_i(B_k, y_k + \varepsilon \cdot a_k)$$

whenever $U_i(B_{\phi(i)}, y_{\phi(i)}) = U_i(B_k, y_k)$.

According to Property (P1) the amounts of money may at most increase until they reach kink point. Indeed, given any agent $i \in N$ and $y_k \in \mathbb{R}$, take the smallest kink point of $f_iB_k$ strictly larger than $y_k$. Then $\varepsilon > 0$ can be chosen such that $y_k + \varepsilon \cdot a_k$ is still smaller than this kink point. Because there are a finite number of agents and a countable number of kink points $\varepsilon > 0$ can be selected such that Property (P1) is satisfied. Furthermore, note that Property (P1) implies for all $i \in N$ that

$$U_i(B_k, y_k + \varepsilon \cdot a_k) = U_i(B_k, y_k) + \varepsilon \cdot a_k \cdot d^+ f_iB_k(y_k)$$

whenever $B_k$ has label $\oplus$.

Property (P2) states that no strong arc arises from a node with label $\ominus$ to a node with label $\oplus$. By the definition of the labels, we have for all $i \in N^{\ominus}$,

$$U_i(B_k, y_k) < U_i(B_{\phi(i)}, y_{\phi(i)})$$

whenever $B_k$ has label $\oplus$.

Using the continuity in money $\varepsilon > 0$ can be selected such that Property (P2) is satisfied.

Finally, if $i \in N^{\oplus}$ and $B_k$ has label $\ominus$ such that $U_i(B_{\phi(i)}, y_{\phi(i)}) > U_i(B_k, y_k)$, Property (P3) implies that

$$U_i(B_{\pi^{\ominus}(i)}, x_{\pi^{\ominus}(i)}) > U_i(B_{\pi^{\ominus}(i)}, y_{\pi^{\ominus}(i)}) \geq U_i(B_k, x_k).$$

So, this property implies that whenever agent $i \in N^{\oplus}$ disliked the bundle $(B_k, y_k)$ with label $\ominus$ in the original situation, he will also dislike or equally appreciate the bundle $(B_k, x_k)$ in the new situation. Note that, again by continuity in money and by (2), $\varepsilon > 0$ can be selected such that Property (P3) is satisfied.
Similarly, we construct a new bijection and money-distribution for the agents in \( N_\oplus \). Define for all \( i, j \in N_\oplus \)

\[
D_{\phi(j)}^{\oplus} := \begin{cases} 
\log(d^{-1}f_{B_{\phi(j)}}(y_{\phi(j)})) & \text{if } U_i(B_{\phi(i)}, y_{\phi(i)}) \leq U_i(B_{\phi(j)}, y_{\phi(j)}), \\
\infty & \text{if } U_i(B_{\phi(i)}, y_{\phi(i)}) > U_i(B_{\phi(j)}, y_{\phi(j)}).
\end{cases}
\]

Note that \( D_{i\phi(i)}^{\oplus} \in \mathbb{R} \) for all \( i \in N_\oplus \) since \( d^{-1}f_{B_{\phi(i)}}(y_{\phi(i)}) > 0 \) for all \( i \in N_\oplus \). So, we can apply Lemma 4.9(ii) with \( I = N_\oplus \), \( J = \varphi(N_\oplus) \) and \( D_d = D_{i\phi(i)}^{\oplus} \) for all for all \( i, j \in N_\oplus \).

Hence, there are real numbers \( \{v_j\}_{j \in \varphi(N_\oplus)} \) and a bijection \( \pi : N_\oplus \rightarrow \varphi(N_\oplus) \) such that \( D_{\pi\oplus}(i) \in \mathbb{R} \) for all \( i \in N_\oplus \) and

\[
D_{\pi\oplus}(i) - v_{\pi\oplus(i)} \geq D_{\pi\oplus}(j) - v_{\pi\oplus(j)} \quad \text{for all } i, j \in N_\oplus.
\] (3)

As \( D_{\pi\oplus}(i) \in \mathbb{R} \) for all \( i \in N_\oplus \), we obtain, by definition of \( D_{\oplus} \) and the fact that there are no circuits with strong arcs, the equality

\[
U_i(B_{\phi(i)}, y_{\phi(i)}) = U_i(B_{\pi\oplus(i)}, y_{\pi\oplus(i)}) \quad \text{for all } i \in N_\oplus.
\] (4)

Indeed, as far as \( \varphi(i) \neq \pi\oplus(i) \) for some \( i \in N_\oplus \), it can be verified (the same argument given below (2) can be applied) that there is a weak arc from node \( (B_{\phi(i)}, y_{\phi(i)}) \) to node \( (B_{\pi\oplus(i)}, y_{\pi\oplus(i)}) \).

The bijection \( \pi : N_\oplus \rightarrow \varphi(N_\oplus) \) and the numbers \( \{v_j\}_{j \in \varphi(N_\oplus)} \) are used to construct a new allocation for the agents in \( N_\oplus \). For all \( i \in N_\oplus \) we define \( b_{\pi\oplus(i)} := e^{-v_{\pi\oplus(i)}} \). Then agent \( i \in N_\oplus \) receives the bundle

\[
(B_{x\oplus(i)}, x_{x\oplus(i)}) \text{ with } x_{x\oplus(i)} := y_{\pi\oplus(i)} - \delta \cdot b_{x\oplus(i)}.
\]

Here, we select, for the moment, \( \delta > 0 \) such that:

(P4) For all \( i \in N \) and every \( 1 \leq k \leq n \) such that node \( B_k \) has label \( \ominus \), the function \( f_{B_k} \) is linear on the interval \([y_k - \delta \cdot b_k, y_k]\).

(P5) For all \( i \in N_\oplus \) and every \( 1 \leq k \leq n \) such that node \( B_k \) has label \( \ominus \), \( \delta > 0 \) satisfies

\[
U_i(B_{\pi\oplus(i)}, y_{\pi\oplus(i)} - \delta \cdot b_{\pi\oplus(i)}) \geq U_i(B_k, y_k),
\]

whenever \( U_i(B_{\phi(i)}, y_{\phi(i)}) > U_i(B_k, y_k) \).

According to Property (P4) the amounts of money may at most decrease until they reach a kink point. A similar reasoning as for Property (P1) can be applied to verify that \( \delta > 0 \) can be selected such that Property (P4) holds. Note that this property implies for all \( i \in N \) that

\[
U_i(B_k, y_k - \delta \cdot b_k) = U_i(B_k, y_k) - \delta \cdot b_k \cdot d^{-1}f_{B_k}(y_k)
\]

whenever \( B_k \) has label \( \ominus \).

Property (P5) states that no weak arc arises from a node with label \( \ominus \) to a node with label \( \ominus \). Similarly as for Property (P2), the definitions of the labels, the continuity in money and (4) guarantee that \( \delta > 0 \) can be selected such that Property (P5) holds.
Finally, if \( i \in N^\ominus \) and \( B_k \) has label \( \ominus \) such that \( U_i(B_{\varphi(i)}, y_{\varphi(i)}) > U_i(B_k, y_k) \), Property (P_6) implies that

\[
U_i(B_{\pi^{\ominus}(i)}, x_{\pi^{\ominus}(i)}) \geq U_i(B_k, y_k) > U_i(B_k, x_k).
\]

So, this property implies that whenever agent \( i \in N^\ominus \) disliked the bundle \( (B_k, y_k) \) with label \( \ominus \) in the original situation, he also will dislike or equally appreciate the bundle \( (B_k, x_k) \) in the new situation. Note that, again by continuity in money and by (4), \( \delta > 0 \) can be selected such that Property (P_3) is satisfied.

Given these new allocations for the agents in \( N^\oplus \) and in \( N^\ominus \), respectively, we now construct the new allocation \( (B_{\pi(i)}, x_{\pi(i)}) \in N \) with the bijection \( \pi \in \Pi \) as follows:

\[
\pi(i) := \begin{cases} 
\pi^{\oplus}(i) & \text{if } i \in N^\oplus, \\
\pi^{\ominus}(i) & \text{if } i \in N^\ominus, \\
\varphi(i) & \text{if } i \in N^\ominus.
\end{cases}
\]

and the money-distribution \( x \in M \) as follows:

\[
x_{\pi(i)} := \begin{cases} 
y_{\pi(i)} + \varepsilon \cdot a_{\pi(i)} & \text{if } i \in N^\oplus, \\
y_{\pi(i)} - \delta \cdot b_{\pi(i)} & \text{if } i \in N^\ominus, \\
y_{\varphi(i)} & \text{if } i \in N^\ominus.
\end{cases}
\]

The numbers \( \delta > 0 \) and \( \varepsilon > 0 \) need to be selected such that they satisfy, next to the Properties (P_1)–(P_6), the following two additional properties:

\textbf{(P_7)} For all \( i \in N^\ominus \) and every \( 1 \leq k \leq n \) such that node \( B_k \) has label \( \ominus \), select \( \delta > 0 \) and \( \varepsilon > 0 \) such that

\[
U_i(B_k, y_k + \varepsilon \cdot a_k) \leq U_i(B_{\pi(i)}, y_{\pi(i)} - \delta \cdot b_{\pi(i)}).
\]

\textbf{(P_8)} \( \varepsilon \cdot \sum_{i \in N^\oplus} a_{\pi(i)} = \delta \cdot \sum_{i \in N^\ominus} b_{\pi(i)}. \)

Property (P_7) states that no strong arc will arise from a node with label \( \ominus \) to a node with label \( \oplus \), when the amounts of money are increased for the agents in \( N^\oplus \) and decreased for the agents in \( N^\ominus \). Note that by definition of the labels and by (4), we have for all \( i \in N^\ominus \) and every \( 1 \leq k \leq n \) such that node \( B_k \) has label \( \ominus \) the strict inequality

\[
U_i(B_k, y_k) < U_i(B_{\varphi(i)}, y_{\varphi(i)}) = U_i(B_{\pi(i)}, y_{\pi(i)}).
\]

So, again from the continuity in money \( \delta > 0 \) and \( \varepsilon > 0 \) can be selected such that Property (P_7) is satisfied.

Property (P_8) guarantees that \( \sum_{k=1}^n x_k = 0 \). After one has selected \( \delta > 0 \) and \( \varepsilon > 0 \), satisfying the Properties (P_1)–(P_7), one can adjust them such that Property (P_8) is also satisfied.
Knowing how the new allocation \((B_{\pi(i)}, x_{\pi(i)})\) is constructed and which properties satisfy, we now prove that the envy-measure of this new allocation is strictly less than the envy-measure of the initial allocation \((B_{\varphi(i)}, y_{\varphi(i)})\). To do so, we need to distinguish four cases:

Case (1). First we show that \(E_i\) has not increased for all agents \(i \in N^\circ\). Take \(i \in N^\circ\).

(i) Let \(1 \leq k \leq n\) such that node \(B_k\) has label \(\circ\). Then, because \(U_i(B_{\pi(i)}, x_{\pi(i)}) = U_i(B_{\varphi(i)}, y_{\varphi(i)})\), it follows that

\[
\sum_{k:B_k \text{ label } \circ} E^k_i(\pi, x) = \sum_{k:B_k \text{ label } \circ} E^k_i(\varphi, y) = 0.
\]

(ii) Let \(1 \leq k \leq n\) such that node \(B_k\) has label \(\odot\). Then

\[
U_i(B_k, x_k) - U_i(B_{\pi(i)}, x_{\pi(i)}) = U_i(B_k, y_k - \delta \cdot b_k) - U_i(B_{\varphi(i)}, y_{\varphi(i)}) < U_i(B_k, y_k) - U_i(B_{\varphi(i)}, y_{\varphi(i)}).
\]

Hence,

\[
\sum_{k:B_k \text{ label } \odot} E^k_i(\pi, x) \leq \sum_{k:B_k \text{ label } \odot} E^k_i(\varphi, y) = 0.
\]

(iii) Let \(1 \leq k \leq n\) such that node \(B_k\) has label \(\oplus\). Then, by Property (P_2) of \(\varepsilon\), it follows that

\[
\sum_{k:B_k \text{ label } \oplus} E^k_i(\pi, x) = 0 = \sum_{k:B_k \text{ label } \oplus} E^k_i(\varphi, y).
\]

By combining equations (5), (6) and (7) it follows that

\[
E_i(\pi, x) \leq E_i(\varphi, y) \quad \text{for all agents } i \in N^\circ.
\]

Case (2). Next, we show that \(E_i\) has not increased for all agents \(i \in N^\circ\). Take \(i \in N^\circ\).

(i) Let \(1 \leq k \leq n\) such that node \(B_k\) has label \(\odot\). Then, by Property (P_3) of \(\delta\), it follows that

\[
\sum_{k:B_k \text{ label } \odot} E^k_i(\pi, x) = 0 = \sum_{k:B_k \text{ label } \odot} E^k_i(\varphi, y).
\]

(ii) Let \(1 \leq k \leq n\) such that node \(B_k\) has label \(\oplus\). If \(U_i(B_k, y_k) < U_i(B_{\varphi(i)}, y_{\varphi(i)})\), then according to Property (P_6) of \(\delta\) we obtain that

\[
U_i(B_k, x_k) < U_i(B_{\pi(i)}, x_{\pi(i)}).
\]

On the other hand, if \(U_i(B_k, y_k) \geq U_i(B_{\varphi(i)}, y_{\varphi(i)})\) then the definition of \(b_{\pi(i)}\) plays an important role. Recall from (3) that
4.5 Proof of Theorem 4.3

\[ D^{\ominus}_{ik} - v_k \geq D^{\ominus}_{i\pi(i)} - v_{\pi(i)}. \]

Recall that \( b_k = e^{-v_k} \), \( b_{\pi(i)} = e^{-v_{\pi(i)}} \), and \( D^{\ominus}_{i\pi(i)} = \log(d^{-f_i}(y_{\pi(i)}) \in \mathbb{R}) \). But since \( U_i(B_k, y_k) \geq U_i(B_{\pi(i)}, y_{\pi(i)}) \), we also have, by definition of \( D^{\ominus} \), that \( D^{\ominus}_{ik} = \log(d^{-f_i}(y_k)) \). Hence, the inequality above yields that

\[ b_k \cdot d^{-f_i}(y_k) \geq b_{\pi(i)} \cdot d^{-f_i}(y_{\pi(i)}). \]

From this inequality it can be derived that

\[
\begin{align*}
U_i(B_k, x_k) - U_i(B_{\pi(i)}, x_{\pi(i)}) &= U_i(B_k, y_k - \delta \cdot b_k) - U_i(B_{\pi(i)}, y_{\pi(i)} - \delta \cdot b_{\pi(i)}) \\
&= U_i(B_k, y_k) - \delta \cdot b_k \cdot d^{-f_i}(y_k) \\
&\quad - \left[ U_i(B_{\pi(i)}, y_{\pi(i)} - \delta \cdot b_{\pi(i)} \cdot d^{-f_i}(y_{\pi(i)}) \right] \\
&= U_i(B_k, y_k) - U_i(B_{\pi(i)}, y_{\pi(i)}) \\
&\quad + \delta \cdot \left[ b_{\pi(i)} \cdot d^{-f_i}(y_{\pi(i)}) - b_k \cdot d^{-f_i}(y_k) \right] \\
&\leq U_i(B_k, y_k) - U_i(B_{\pi(i)}, y_{\pi(i)}) \\
&= U_i(B_k, y_k) - U_i(B_{\phi(i)}, y_{\phi(i)}).
\end{align*}
\]

The last equality follows from (4). Hence, we can conclude that

\[ \sum_{k: B_k \text{ label } \ominus} E^k_i(\pi, x) \leq \sum_{k: B_k \text{ label } \ominus} E^k_i(\varphi, y). \tag{9} \]

(iii) Let \( 1 \leq k \leq n \) such that node \( B_k \) has label \( \ominus \). Then, by Property \((P_\tau)\) of \( \delta \) and \( \varepsilon \), it follows that

\[ \sum_{k: B_k \text{ label } \ominus} E^k_i(\pi, x) = 0 = \sum_{k: B_k \text{ label } \ominus} E^k_i(\varphi, y). \tag{10} \]

By combining equations (8), (9) and (10) it follows that

\[ E_i(\pi, x) \leq E_i(\varphi, y) \text{ for all agents } i \in N^{\ominus}. \]

Case (3). Now, it is shown that \( E_i \) has not increased for all agents \( i \in N^{\ominus} \). Take \( i \in N^{\ominus} \).

(i) Let \( 1 \leq k \leq n \) such that node \( B_k \) has label \( \ominus \). Then, since \( x_k = y_k \) and \( x_{\pi(i)} > y_{\pi(i)} \), we have that

\[
\begin{align*}
U_i(B_k, x_k) - U_i(B_{\pi(i)}, x_{\pi(i)}) &< U_i(B_k, y_k) - U_i(B_{\pi(i)}, y_{\pi(i)}) \\
&= U_i(B_k, y_k) - U_i(B_{\phi(i)}, y_{\phi(i)}).
\end{align*}
\]

The last equality follows from (2). Hence,

\[ \sum_{k: B_k \text{ label } \ominus} E^k_i(\pi, x) \leq \sum_{k: B_k \text{ label } \ominus} E^k_i(\varphi, y). \tag{11} \]
(ii) Let \(1 \leq k \leq n\) such that node \(B_k\) has label \(\odot\). Then, since \(x_k < y_k\) and \(x_{\pi(i)} > y_{\pi(i)}\), we have that
\[
U_i(B_k, x_k) - U_i(B_{\pi(i)}, x_{\pi(i)}) < U_i(B_k, y_k) - U_i(B_{\pi(i)}, y_{\pi(i)}) = U_i(B_k, y_k) - U_i(B_{\varphi(i)}, y_{\varphi(i)}).
\]
Again, the last equality follows from (2). Hence,
\[
\sum_{k:B_k \text{ label } \odot} E^k_i(\pi, x) \leq \sum_{k:B_k \text{ label } \odot} E^k_i(\varphi, y). \tag{12}
\]

(iii) Let \(1 \leq k \leq n\) such that node \(B_k\) has label \(\oplus\). If \(U_i(B_k, y_k) < U_i(B_{\varphi(i)}, y_{\varphi(i)})\), then according to Property (P3) of \(\varepsilon\) we obtain that
\[
U_i(B_k, x_k) < U_i(B_{\pi(i)}, x_{\pi(i)}).
\]
On the other hand, if \(U_i(B_k, y_k) \geq U_i(B_{\varphi(i)}, y_{\varphi(i)})\) then the definition of \(a_{\pi(i)}\) plays an important role. Recall from (1) that
\[
D_{ik}^\oplus - v_k \leq D_{i\pi(i)}^\oplus - v_{\pi(i)}.
\]
Recall that \(a_k = e^{-v_k}\), \(a_{\pi(i)} = e^{-v_{\pi(i)}}\) and \(D_{i\pi(i)}^\oplus = \log(d_i f_i B_{\pi(i)}(y_{\pi(i)}))\) (since \(D_{i\pi(i)}^\oplus \in \mathbb{R}\)). But since \(U_i(B_k, y_k) \geq U_i(B_{\varphi(i)}, y_{\varphi(i)})\), we also have, by definition of \(D^\oplus\), that \(D_{ik}^\oplus = \log(d_i f_i B_k(y_k))\). Hence, the inequality above yields that
\[
ak_k \cdot d_i f_i B_k(y_k) \leq a_{\pi(i)} \cdot d_i f_i B_{\pi(i)}(y_{\pi(i)}).
\]
From this inequality it can be derived that
\[
U_i(B_k, x_k) - U_i(B_{\pi(i)}, x_{\pi(i)}) = U_i(B_k, y_k + \varepsilon \cdot a_k) - U_i(B_{\pi(i)}, y_{\pi(i)} + \varepsilon \cdot a_{\pi(i)})
\]
\[
= U_i(B_k, y_k) - U_i(B_{\pi(i)}, y_{\pi(i)}) + \varepsilon \cdot [a_k \cdot d_i f_i B_k(y_k) - a_{\pi(i)} \cdot d_i f_i B_{\pi(i)}(y_{\pi(i)})]
\]
\[
\leq U_i(B_k, y_k) - U_i(B_{\pi(i)}, y_{\pi(i)}) = U_i(B_k, y_k) - U_i(B_{\varphi(i)}, y_{\varphi(i)}).
\]
The last equality follows from (2). Hence, we can conclude that
\[
\sum_{k:B_k \text{ label } \oplus} E^k_i(\pi, x) \leq \sum_{k:B_k \text{ label } \oplus} E^k_i(\varphi, y). \tag{13}
\]
By combining equations (11), (12) and (13) it follows that
\[
E_i(\pi, x) \leq E_i(\varphi, y)
\]
for all agents \(i \in N^\oplus\).

Case (4). Finally, we show that \(E_i\) has strictly decreased for the agent \(i^* \in N^\oplus\) to whom the bundle \((B_{k^*}, y_{k^*})\) is assigned by \(\varphi\).
4.5 Proof of Theorem 4.3

(i) Recall that in the envy-graph there was a strong arc from node $B_{k^*}$ to node $B_{l^*}$. This means that

$$U_{i^*}(B_{\varphi(i^*)}, y_{\varphi(i^*)}) = U_{i^*}(B_{k^*}, y_{k^*}) < U_{i^*}(B_{l^*}, y_{l^*})$$

and thus $E_{i^*}^k(\varphi, y) > 0$. Furthermore,

$$U_{i^*}(B_{l^*}, x_{l^*}) = U_{i^*}(B_{\pi(i^*)}, x_{\pi(i^*)})$$

$$= U_{i^*}(B_{l^*}, y_{l^*} - \delta \cdot b_{l^*}) - U_{i^*}(B_{\pi(i^*)}, y_{\pi(i^*)} + \varepsilon \cdot a_{\pi(i^*)})$$

$$< U_{i^*}(B_{l^*}, y_{l^*}) - U_{i^*}(B_{\pi(i^*)}, y_{\pi(i^*)})$$

$$= U_{i^*}(B_{l^*}, y_{l^*}) - U_{i^*}(B_{\varphi(i^*)}, y_{\varphi(i^*)}).$$

The last equality follows from (2) since $i^* \in N^\oplus$. Combining the strict inequality above with the observation that $E_{i^*}^k(\varphi, y) > 0$ yields,

$$E_{i^*}^k(\pi, x) = [U_{i^*}(B_{l^*}, x_{l^*}) - U_{i^*}(B_{\pi(i^*)}, x_{\pi(i^*)})]_+$$

$$< U_{i^*}(B_{l^*}, y_{l^*}) - U_{i^*}(B_{\varphi(i^*)}, y_{\varphi(i^*)})$$

$$= E_{i^*}^k(\varphi, y). \quad (14)$$

(ii) We have already shown (Case (3)) that

$$E_{i^*}^k(\pi, x) \leq E_{i^*}^k(\varphi, y) \quad \text{for all } 1 \leq k \leq n \text{ with } k \neq l^*.$$

By combining equations (14) and (15) it follows that

$$E_{i^*}(\pi, x) < E_{i^*}(\varphi, y) \quad \text{for agent } i^* \in N^\oplus.$$

Hence, we can conclude that

$$E(\pi, x) = \sum_{i \in N^\oplus} E_i(\pi, x) + \sum_{i \in N^\oplus} E_i(\pi, x) + \sum_{i \in N^\oplus \setminus i^*} E_i(\pi, x) + E_{i^*}(\pi, x)$$

$$< \sum_{i \in N^\oplus} E_i(\varphi, y) + \sum_{i \in N^\oplus} E_i(\varphi, y) + \sum_{i \in N^\oplus \setminus i^*} E_i(\varphi, y) + E_{i^*}(\varphi, y)$$

$$= E(\varphi, y).$$

So, the envy-measure of the new allocation $(B_{\varphi(i)}, x_{\varphi(i)})_{i \in N}$ is strictly less than the envy-measure of the initial allocation $(B_{\varphi(i)}, y_{\varphi(i)})_{i \in N}$. \qed

Remark. In case the utility functions are quasi-linear, the method described in the proof of Proposition 4.10 leads to the algorithm given in Klijn (2000). This algorithm constructs from every initial $n$-partition a bijection and a money-distribution such that the associated allocation is envy-free.

Given the results of Proposition 4.2 and Proposition 4.10, the proof of Theorem 4.3K is now straightforward.
Proof of Theorem 4.3. Let \( \{B_k\}_{1 \leq k \leq n} \) be any \( n \)-partition of \( \Omega \). Take \( \pi \in \Pi \) and \( x \in \mathcal{M} \) and define \( \varepsilon := E(\pi, x) \). Then the set \( \mathcal{A}_\varepsilon = \{ y \in \mathcal{M} \mid E(\pi, y) \leq \varepsilon \} \) is non-empty. Moreover, Proposition 4.2 tells us that it is compact and therefore there exists a money-distribution \( x^\pi \in \mathcal{A}_\varepsilon \) such that \( E(\pi, x^\pi) \leq E(\pi, y) \) for all \( y \in \mathcal{A}_\varepsilon \). Observe that this latter inequality is also true for \( y \notin \mathcal{A}_\varepsilon \). Hence, we can conclude that for every \( \pi \in \Pi \) there exists a money-distribution \( x^\pi \in \mathcal{M} \) such that
\[
E(\pi, x^\pi) \leq E(\pi, y) \quad \text{for all } y \in \mathcal{M}.
\]
Suppose \( E(\pi, x^\pi) > 0 \) for all \( \pi \in \Pi \). Then \( E(\varphi, x^\varphi) := \min\{E(\pi, x^\pi) \mid \pi \in \Pi\} > 0 \). By Proposition 4.10 there exists a bijection \( \sigma \in \Pi \) and a money-distribution \( z \in \mathcal{M} \) such that
\[
E(\sigma, z) < E(\varphi, x^\varphi) \leq E(\pi, x^\pi) \quad \text{for all } \pi \in \Pi.
\]
So, in particular \( E(\sigma, z) < E(\sigma, x^\sigma) \). This contradicts the definition of \( x^\sigma \).

Hence, there exists a bijection \( \pi \in \Pi \) such that \( E(\pi, x^\pi) = 0 \) and this means that the allocation \((B_{\pi(i)}, x^\pi_{\pi(i)})_{i \in \mathcal{N}}\) is envy-free. This completes the proof. \( \Box \)
The Nucleolus of Trees with Revenues

5.1 Introduction

The chapter studies the nucleolus of a tree with revenues. Trees with revenues generalize standard cost trees. Here, players do not only pay for their connections to the root, but a player can also earn some revenue from being connected to the root. The results of this chapter can also be found in Meertens and Potters (2004) and in Meertens and Potters (2005).

Certain cost allocation problems can be modeled adequately by standard tree networks, i.e., a rooted tree with non-negative costs on the edges and a number of players in each node. A typical example is a cable network connecting several villages (the nodes of the tree) with a central supplier (the root). Some of the villages are directly connected to the supplier and others indirectly. The cables have certain maintenance costs which have to be allocated among the users of the network (the inhabitants of the villages). The cost allocation problem that arises, is to determine which amount each of the inhabitants should contribute to the total maintenance costs.

A natural approach to solve this problem is to model it as a cooperative TU-game and use a solution concept to derive a cost allocation. In Littlechild (1974) and in Littlechild and Thompson (1977) this is done in the case that the underlying tree is a line graph. They study these so-called airport problems in a cooperative game theory framework and derive a simple algorithm for calculating its nucleolus (Schmeidler (1969)). The first algorithm for calculating the nucleolus of an arbitrary tree, a so-called standard tree, is due to Megiddo (1978). It was modified by Galil in 1980. Maschler, Reijnierse and Potters (1995) introduced yet another algorithm and Granot, Maschler, Owen and Zhu (1996) provide a characterization of the nucleolus of a standard tree game.

Meanwhile, Littlechild and Owen (1977) introduced the idea of revenues in an airport prob-
lem. Their motivation is that cost-games derived from airport problems form only a partial representation of the actual situation. They do not take into account the revenues or other benefits generated by aircraft movements. Consequently, the authors introduce airport profit games by defining the value of a coalition to be the maximal net benefit this coalition can generate by connecting (some of) its members. It is also possible for a coalition to refrain from connecting anybody. Littlechild and Owen claim that Littlechild’s algorithm (1974) for the nucleolus of airport cost-games remains valid for computing the nucleolus of an airport profit game. However, this statement is incorrect (unless the revenue of a player is at least as much as his individual costs). This observation is made in Brânzei, Iñárra, Tijs and Zarzuelo (2003). The authors demonstrate, by means of an example, that Littlechild’s algorithm is no longer suitable for calculating the nucleolus of airport profit games. In addition they provide a new polynomial time algorithm for computing the nucleolus for these type of games.

This chapter follows the approach taken by Brânzei et al. (2003) to study trees with revenues. Here, a rooted tree with non-negative costs is given in which players can earn a (non-negative) revenue from being connected to the root. For example, one can think of the root as an internet-provider and the players as inhabitants of villages who can earn some money via the internet, if they have a connection. A cooperating coalition of players is assumed to connect those members to the root who together yield the highest net profit (i.e., the maximum of their total revenues minus the construction costs). Again, it is also possible for a coalition to refrain from connecting anybody to the root. By assigning to each coalition the maximal profit as its value, we obtain the TU-game corresponding to the tree with revenues.

The focus is on the nucleolus of this TU-game and we present in Subsection 5.3.2 a polynomial time algorithm to compute it. The algorithm considers a finite sequence of trees, starting with the original tree and where each tree is obtained of the preceding one, either by a transformation or by Davis and Maschler reduction (Davis and Maschler (1965)). These transformations as well as the effects on the nucleolus are discussed in Subsection 5.2.4. They provide us standard trees with revenues. Given such a standard tree, the maximal excess at the nucleolus is determined by four different types of numbers, each defined by the revenues and/or costs of the tree. The minimum among these numbers provides us those coalitions with maximal excess and these coalitions help us to compute the nucleolus payoff of some players. Reducing these players from the tree and using the result that the nucleolus of a (standard) tree with revenues satisfies the reduced game property (Sobolev (1975)), yields that the nucleoli of the trees in the sequence can be converted into the nucleolus of the original tree.

In this chapter we also give some monotonicity properties satisfied by the nucleolus and we give a characterization for the nucleolus as a solution rule for trees with revenues. Usually, the nucleolus is not too good in following monotonicity rules. It may happen, for example, that by increasing the value of a single coalition, some members of that coalition receive less as nucleolus payoff. This lack of monotonicity is demonstrated, by means of a convex TU-game with four players, in Hokari (2000) (see also Megiddo (1974) and Maschler (1992) for examples and a discussion on this issue). However, Maschler et al. (1995) prove that the nucleolus of a standard cost tree is cost-monotonic (i.e., no player benefits whenever the costs of an edge increases). Furthermore, they prove that the nucleoli of all subgames of a standard
cost tree form a population monotonic allocation scheme (pmas), a concept introduced by Thomson (1983) (see also Sprumont (1990)). Roughly speaking, a pmas is a table which contains for each coalition a payoff vector such that every player gets a weakly higher payoff in larger coalitions. In Subsection 5.3.3 we show that also for the nucleolus of a tree with revenues these two monotonicity properties apply. Since, we consider in this chapter trees in which players may earn some revenue, it seems also natural to ask whether the nucleolus is revenue-monotonic (i.e., the nucleolus payoff for each player weakly increases whenever the revenue of a player increases). In Subsection 5.3.3 this question is answered affirmatively.

In Subsection 5.3.4 we give a characterization of the nucleolus as a solution rule for trees with revenues via seven properties, among which reasonableness (Sudhölter (1997)), consistency (Potters and Sudhölter (1999) call this property \(\nu\)-consistency) and cost-monotonicity are the more important ones. Reasonableness (on both sides) states that each player makes a non-negative contribution to the costs of the tree and it states that this contribution does not exceed the total costs of the path connecting him with the root, a player’s stand-alone costs (Moulin and Shenker (1992)). A solution rule is consistent when a player leaves the tree with a reasonable payoff and contributes to the costs accordingly (by diminishing the costs of his path to the root), then the payoff according the solution for the remaining players remains the same. It turns out that two solution rules satisfying reasonableness, consistency and cost-monotonicity coincide whenever they coincide for trees with two players. So, due to these three properties of a solution rule we may restrict our attention to trees with two inhabitants.

To characterize the nucleolus of a tree with two inhabitants we need four additional properties. Profit-making states that whenever the inhabitants of a leaf generate a non-positive net benefit, they will receive no payoff and the payoff to the other players is indifferent whether this leaf is present or not. So, in this property it is assumed that the players will refrain from connecting such a leaf with the root, since it does not generate any profit. Therefore, the inhabitants are not entitled to any part of the total profit, nor are they willing to contribute anything to the costs. According to the property of minimal obligation, each player should at least pay his marginal costs. This means that if a player is the only inhabitant of a leaf, he should at least pay the costs connecting him to its predecessor, before he is entitled to receive any payoff. On the other hand, the property of minimal rights first states that whenever the revenue of a player exceeds his stand-alone costs, he is entitled to receive at least the positive difference. Finally, the property of indispensability states that an inhabitant of a leaf with a revenue exceeding the net benefit of the leaf should at least receive as much payoff as the other inhabitants. Such a player can be seen as indispensable, in the sense that without him the leaf would have a net benefit less than or equal to zero. The main result of Subsection 5.3.4 is that these seven properties characterize the nucleolus as a solution rule for trees with revenues.

Finally in Subsection 5.3.5, we consider an example of a tree with revenues and use the algorithm, as presented in Subsection 5.3.2, to compute its nucleolus. But, let us start by fixing the terminology used in this chapter.
5.2 The tree with revenues and the associated game

5.2.1 Definitions and notations

In this chapter we consider trees with revenues \( \Gamma := (\{V, E, r\}, \{N_p, c_p\}_{p \in V_0}, (b_i)_{i \in N}) \) which have the following features:

- \( \{V, E, r\} \) is a tree with a root, in which \( V \) is a finite set of nodes and \( E \) is the set of edges. We denote the root by \( r \) and write \( V_0 := V \setminus \{r\} \). If \( p \in V_0 \), then there is a unique path \( p_0 = r, p_1, \ldots, p_{k-1}, p_k = p \) (with \( (p_{i-1}, p_i) \in E \) for all \( 0 \leq i \leq k \)) from the root to \( p \). The predecessor of \( p \), denoted by \( \pi(p) \), is by definition \( p_{k-1} \) and the children of \( p \), denoted by \( Ch(p) \), are the nodes \( q \in V_0 \) with \( \pi(q) = p \). Furthermore, we denote \( q \preceq p \) if node \( q \) is on this path, i.e., \( q = p_\ell \) for a certain number \( 0 \leq \ell \leq k \).

- Each edge has a cost defined by the map \( c : E \rightarrow \mathbb{R}_+ \). We write \( c_p \in \mathbb{R}_+ \) for the cost of the edge \( e_p := (\pi(p), p) \in E \) (with the convention that \( c_r := 0 \)).

- In every node \( p \in V \) there is a finite set of players (or inhabitants) \( N_p \). The collection of sets \( \{N_p\}_{p \in V} \), with the assumption that \( N_p \cap N_q = \emptyset \) whenever \( p \neq q \), is the population distribution. We denote \( N := \bigcup_{p \in V} N_p \).

- Each player \( i \in N \) has a non-negative revenue \( b_i \).

Given a tree with root \( \{V, E, r\} \), a node \( p \in V_0 \) is a leaf if there exist no nodes \( q \in V_0 \) with \( p = \pi(q) \). A non-empty set \( T \subseteq V \) is a trunk if \( q \in T \) whenever \( p \in T \) and \( q \preceq p \). This means, in particular, that \( \{r\} \subseteq T \). Moreover, a connected set of nodes \( B \subseteq V \) is a branch if \( q \in B \) whenever \( p \in B \) and \( q \preceq p \). Observe that a branch contains a node \( p \in V \) such that \( B = \{q \in V \mid q \preceq p \} \). We call this branch \( B_p \). The children \( Ch(T) \) of a trunk \( T \) are those nodes \( p \notin T \) such that \( \pi(p) \in T \). If a trunk has exactly one child \( p \), we call it a maximal trunk and use the notation \( T_p \). Observe that a maximal trunk \( T_p \) does not contain node \( p \) and furthermore, note that

\[
T = V \setminus \bigcup_{p \in Ch(T)} B_p = \bigcap_{p \in Ch(T)} T_p.
\]

With every tree with revenues \( \Gamma \) we associate a cooperative TU-game. Let \( S \subseteq N \) be any coalition of players. Then this coalition can generate the total revenues \( b(S) := \sum_{i \in S} b_i \) when all members are connected to the root. The minimal costs to connect all players in \( S \) to the root are the costs of the trunk \( T(S) \) defined by

\[
T(S) := \{q \in V \mid \text{there is a node } p \in V \text{ with } q \preceq p \text{ and } N_p \cap S \neq \emptyset \}.
\]

So, the trunk \( T(S) \) is the smallest trunk containing all nodes with at least one member of \( S \). Observe that the trunk \( T(\{i\}) \) for a certain player \( i \in N_p \) consists of all nodes on the path from node \( p \) to the root. Given the trunk \( T(S) \) of coalition \( S \subseteq N \), the costs are

\[
c_T(S) := c(T(S)) = \sum_{p \in T(S)} c_p.
\]

Hence, a coalition \( S \subseteq N \) may earn
5.2 The tree with revenues and the associated game

\[ w_T(S) := b(S) - c_T(S). \]

However, it may happen that a coalition can generate a higher profit, by not connecting everybody to the root. Therefore, it seems reasonable that a coalition connects those members to the root which yield the highest profit. Consequently, we define for a tree with revenues \( \Gamma \) the corresponding TU-game \( \langle N, v_T \rangle \) by

\[ v_T(S) := \max_{R \subseteq S} w_T(R) \text{ for all } S \subseteq N. \]

So, the value \( v_T(S) \) for coalition \( S \subseteq N \) represents the maximal profit coalition \( S \) may attain. If a coalition \( S \) attains this profit by connecting all members, i.e., \( v_T(S) = w_T(S) \), then coalition \( S \) is called effective. Since it is also possible for a coalition to connect nobody, the TU-game \( \langle N, v_T \rangle \) is non-negative. The following proposition states that it is convex (see Section 2.2 for the definition).

**Proposition 5.1.** The TU-game \( \langle N, v_T \rangle \) is convex.

**Proof.** Let \( S_1, S_2 \subseteq N \), then

\[
\begin{align*}
c_T(S_1) + c_T(S_2) &= c(T(S_1)) + c(T(S_2)) \\
&= c(T(S_1) \cup T(S_2)) + c(T(S_1) \cap T(S_2)) \\
&\geq c(T(S_1 \cup S_2)) + c(T(S_1 \cap S_2)) \\
&= c_T(S_1 \cup S_2) + c_T(S_1 \cap S_2).
\end{align*}
\]

The inequality can be derived from the fact that \( T(S_1 \cup S_2) = T(S_1) \cup T(S_2) \) and that \( T(S_1 \cap S_2) \subseteq T(S_1) \cap T(S_2) \). From this inequality above, it immediately follows that the TU-game \( \langle N, w_T \rangle \) is convex. Since, \( v_T(S) = \max_{R \subseteq S} w_T(R) \) for all \( S \subseteq N \), it is straightforward to verify that the TU-game \( \langle N, v_T \rangle \) is also convex. \( \square \)

Since convexity of a TU-game implies (total) balancedness, Proposition 5.1 yields in particular for the TU-game \( \langle N, v_T \rangle \) to have a non-empty core \( C(v_T) \) (see Section 2.2 for all the definitions). The main goal of this chapter is to study for this class of TU-games a particular core element, namely, the nucleolus (Schmeidler (1969)). Before we do so, we first state the formal definition of the nucleolus and repeat several results regarding the nucleolus of a convex TU-game which will turn out to be useful in our analysis. This is done in the next subsection.

5.2.2 The nucleolus of a TU-game

In Section 2.2 and Section 2.3 several set-valued solution concepts for TU-games are discussed. In this subsection we repeat an other well-known solution concept, the nucleolus.

Let \( \langle N, v \rangle \) be a TU-game with a non-empty imputation set \( I(v) \). Denote \( B := \{ S \subseteq N \mid \emptyset \neq S \subseteq N \} \) for the collection of proper subsets of \( N \). Given \( x \in I(v) \), the vector \( \langle \text{Exc}(S, x) \rangle_{S \in B} \in \mathbb{R}^B \) denotes for every coalition \( S \in B \) its excess in imputation \( x \in I(v) \), i.e., \( \text{Exc}(S, x) := v(S) - x(S) \).
The excess of a coalition in an imputation can be understood as the dissatisfaction of this coalition with the proposed payoff vector. The nucleolus is obtained by performing the following sequence of minimizations. First, identify the imputations at which the excess of the most dissatisfied coalition is the smallest. Then among the minimizers, identify the imputations at which the excess of the second most dissatisfied coalition is the smallest, and so on. The following definition gives a more formal statement on the notion of the nucleolus.

**Definition (Schmeidler (1969)).** The nucleolus of \((N, v)\) is defined as,

\[
\nu(v) := \{ x \in \mathcal{I}(v) \mid \theta \circ \text{Exc}(S, x)_{S \in \mathcal{B}} \leq \theta \circ \text{Exc}(S, y)_{S \in \mathcal{B}} \text{ for all } y \in \mathcal{I}(v) \},
\]

in which the map \(\theta : \mathbb{R}^{|\mathcal{B}|} \rightarrow \mathbb{R}^{|\mathcal{B}|}\) orders the coordinates of a vector in a weakly decreasing order and in which \(\leq\) denotes the lexicographic order on \(\mathbb{R}^{|\mathcal{B}|}\).

For a TU-game \((N, v)\) with a non-empty imputation set the nucleolus is proven to be a singleton. Moreover, it is contained in the union of the kernel with the core (Schmeidler (1969)). Observe that the latter statement in particular implies that the nucleolus is contained in the reactive bargaining set (Theorem 2.3). So, the nucleolus is the unique imputation that lexicographically minimizes the vector of non-increasingly ordered excesses over the set of imputations. For simplicity, we write \(\nu\) instead of \(\nu(v)\) and we denote

\[
\mathcal{D}_1(\nu) := \{ S \in \mathcal{B} \mid \text{Exc}(S, \nu) \geq \text{Exc}(R, \nu) \text{ for all } R \in \mathcal{B} \},
\]

i.e., the set of proper subsets of \(N\) with maximal excess in \(\nu\). Note that if the TU-game \((N, v)\) has a non-empty core, the maximal excess in the nucleolus is at most zero. Therefore, we write

\[
E(\nu) := -\max_{S \in \mathcal{B}} \text{Exc}(S, \nu).
\]

A drawback of the nucleolus is that for most TU-games it is difficult to compute. It requires an exponential number of computations since the solution must adhere to coalitional conditions. Thus, if one attempts to compute the nucleolus by simply following its definition, it would take an exponential amount of time. In the literature, several algorithms can be found for computing the nucleolus (see e.g., Kopelowitz (1967), Maschler, Peleg and Shapley (1979) and Potters, Reijnierse and Ansing (1995)). They all require solutions of a series of linear programs. In several cases one can nonetheless bypass all the computational complexity and compute the nucleolus more easily, i.e., there are several classes of games for which one can compute the nucleolus in polynomial time (see Granot, Granot and Zhu (1998) for an overview). One of these classes is the class of TU-games derived from trees with revenues as described in the previous subsection. This is demonstrated in the forthcoming sections of this chapter.

In the remainder of this subsection we repeat two results regarding the nucleolus of a convex TU-game. The first result states that if the TU-game is also non-negative, then a player receives zero under the nucleolus if and only if he is a null-player (i.e., he does not have any contribution in a coalition).

**Lemma 5.2.** If \((N, v)\) is a non-negative, convex TU-game and \(v\) is the nucleolus, then \(v_i = 0\) if and only if \(v(N) = v(N \setminus \{i\})\).
5.2 The tree with revenues and the associated game

Proof. \( \Rightarrow \) Since the TU-game \( (N, v) \) is convex, the nucleolus is contained in the core. Thus, since \( (N, v) \) is non-negative, \( v_i = 0 \) for a certain player \( i \in N \) implies that \( v\{i\} = 0 \). Hence, \( s_{ij}(\nu) = 0 \) for all \( j \in N \setminus \{i\} \) (see Section 2.3 for the definition). Since \( \nu \) is also an element of the kernel, it holds that \( s_{ji}(\nu) = 0 \) for all \( j \in N \setminus \{i\} \). This means that for every \( j \in N \setminus \{i\} \) there exists a coalition \( S_j \subseteq N \setminus \{i\} \) with \( j \in S_j \) such that

\[
\nu(S_j) - \nu(S_j) = 0.
\]

Let \( S \subseteq N \setminus \{i\} \) be a coalition with \( v(S) = \nu(S) \) such that \( |S| \) is maximal. Suppose \( S \neq N \setminus \{i\} \). Then there exists a player \( j \in N \setminus S \) with \( j \neq i \). Convexity of \( (N, v) \) tells us that

\[
0 = v(S_j) - \nu(S_j) + v(S) - \nu(S) \leq v(S_j \cup S) - \nu(S_j \cup S) + v(S_j \cap S) - \nu(S_j \cap S) \leq 0.
\]

The last inequality follows from the fact that \( \nu \in C(v) \). So, we can conclude that \( v(S_j \cup S) = \nu(S_j \cup S) \). But this contradicts the maximality of \( |S| \).

Hence, \( v(N \setminus \{i\}) = \nu(N \setminus \{i\}) \). By using the efficiency of \( \nu \) and the assumption that \( \nu_i = 0 \), it follows that \( v(N) = \nu(N \setminus \{i\}) \).

\( \Leftarrow \) The converse is straightforward and left to the reader. \( \square \)

The following result is taken from Arin and Iñarrea (1998) and is stated here without a proof. It characterizes which coalitions attain the maximal excess in the nucleolus of a convex game. Their result will be very useful in our analysis. If \( \{S_1, \ldots, S_k\} \) is a partition of \( N \), then the family \( \{N \setminus S_1, \ldots, N \setminus S_k\} \) formed by its complements is an antipartition of \( N \).

Theorem 5.3 (Arin and Iñarrea (1998)). If \( \nu \) is the nucleolus of a convex TU-game, then \( D_1(\nu) \) contains a partition or an antipartition of \( N \). \( \square \)

5.2.3 The reduced tree with revenues and the associated game

Let \( \Gamma \) be a tree with revenues. This subsection describes the reduced tree with revenues with respect to a player and a reasonable payoff.

Assume that the players in the grand coalition \( N \) have decided that some inhabitant of node \( p \in V \), say player \( i \in N_p \), receives a payoff \( z \in \mathbb{R}_+ \). It is reasonable to assume that \( z \leq b_i \) as player \( i \) uses the edges in the tree to obtain his revenue \( b_i \). The difference \( b_i - z \) can be understood as player \( i \)’s contribution to the total costs of the tree. However, it is equally reasonable to assume that this contribution should not exceed the total costs \( c_\Gamma(\{i\}) \) of the path \( T(\{i\}) \), the set of edges needed to connect him with the root (player \( i \)’s stand-alone costs). In case \( z \) equals the payoff of player \( i \) in a core allocation of \( (N,v_\Gamma) \), these two assumptions will be satisfied. This results from the following lemma. Recall that \( [a]_+ \) is an abbreviation for \( \max\{a,0\} \).

Lemma 5.4. If \( x \in C(v_\Gamma) \), then \( [b_i - c_\Gamma(\{i\})]_+ \leq x_i \leq b_i \) for all \( i \in N \).
The following proposition states that the TU-game of a reduced tree is the reduced game of Proposition 5.5.

Since, \( x_i \geq [b_i - c_T(\{i\})]_+ \). Furthermore, note that for all \( R \subseteq N \setminus \{i\} \),

\[
w_T(R \cup \{i\}) = b(R) + b_i - c_T(R \cup \{i\}) \leq b(R) + b_i - c_T(R) = w_T(R) + b_i.
\]

Hence,

\[
v_T(N) = \max \left\{ \max_{R \subseteq N \setminus \{i\}} w_T(R \cup \{i\}), \max_{R \subseteq N \setminus \{i\}} w_T(R) \right\}
\]

\[
\leq \max_{R \subseteq N \setminus \{i\}} w_T(R) + b_i = v_T(N \setminus \{i\}) + b_i.
\]

Since, \( x_i \leq v_T(N) - v_T(N \setminus \{i\}) \), the inequality above yields that \( x_i \leq b_i \).

From Lemma 5.4 we can conclude that \( 0 \leq b_i - x_i \leq c_T(\{i\}) \) for all \( i \in N \), whenever \( x \in C(v_T) \). So, there exist imputations \( x \in \mathcal{I}(v_T) \) for which \( 0 \leq b_i - x_i \leq c_T(\{i\}) \) for all \( i \in N \). Since, in such an imputation the contribution of player \( i \) is non-negative and it does not exceed the costs of the path from node \( p \) to the root, the difference \( b_i - x_i \) can be used to cover a part of the costs of this path \( T(\{i\}) \). The question which part of these costs should be covered, allows for different answers and each of them leads to a different kind of reduced tree (see e.g., Potters and Sudhölter (1999)). Here, we assume that the amount \( b_i - x_i \) is used to reduce the costs of the path \( T(\{i\}) \) starting with the edge \( e_p \), i.e., player \( i \) first contributes to the costs \( c_p \) and if this contribution exceeds \( c_p \) even to the costs \( c_{\pi(p)} \) and so on.

Knowing how to reduce the costs, we can formalize the idea of a reduced tree with respect to a player and his payoff. Take \( i \in N \), say \( i \in N_p \) for a certain node \( p \in V \) and let \( x \in \mathcal{I}(v_T) \) such that \( 0 \leq b_i - x_i \leq c_T(\{i\}) \) for all \( i \in N \) (e.g., \( x \in C(v_T) \)). The reduced tree with revenues of \( \Gamma \) with respect to \( i \) and his payoff \( x_i \) is the tree with revenues

\[
\Gamma^{-i}_x := \langle \{V, E, r\}, \{N'_q, c'_q\}_{q \in V_0}, (b_j)_{j \in N \setminus \{i\}} \rangle.
\]

Here, \( N'_q := N_p \setminus \{i\} \), \( N'_q := N_q \) for all \( q \neq p \) and the costs \( \{c'_q\}_{q \in V_0} \) are defined by:

\[
c'_q := c_q \quad \text{for all } q \notin T(\{i\}),
\]

\[
c'_q := [\min\{c_q, c(B_q \cap T(\{i\})) - (b_i - x_i)\}]_+ \quad \text{for all } q \in T(\{i\}).
\]

Since, \( \Gamma^{-i}_x \) is again a tree with revenues it generates a TU-game. Next, we show that whenever \( x \in C(v_T) \) this TU-game is in fact the Davis and Maschler reduced game (Davis and Maschler (1965)) of \( \langle N, v_T \rangle \) with respect to \( i \) and \( x \). This reduced game of \( \langle N, v_T \rangle \), with respect to \( i \) and \( x \), is the TU-game \( \langle N \setminus \{i\}, v^{-i,x}_T \rangle \) defined by

\[
v^{-i,x}_T(S) := \begin{cases} 
v_T(N) - x_i & \text{if } S = N \setminus \{i\}, \\ 
\max\{v_T(S), v_T(S \cup \{i\}) - x_i\} & \text{if } S \neq \emptyset, N \setminus \{i\}, \\ 
0 & \text{if } S = \emptyset.
\end{cases}
\]

The following proposition states that the TU-game of a reduced tree is the reduced game of the original tree. A proof can be found in Section 5.4.

**Proposition 5.5.** Let \( i \in N \) and \( x \in C(v_T) \), then

\[
v^{-i,x}_T(S) = v^{-i,x}_T(S) \text{ for all } S \subseteq N \setminus \{i\}.
\]
Remark. The nucleolus of a convex TU-game satisfies the reduced game property (Sobolev (1975), see also Peleg (1986)). Because the TU-game \((N, v_T)\) is convex (Proposition 5.1), this means that if \(\nu\) is the nucleolus of \((N, v_T)\), then \((\nu_j)_{j \neq i}\) is the nucleolus of the reduced game with respect to \(i\) and \(\nu\). Furthermore, this (reduced) nucleolus is a core allocation of the reduced game (because this reduced game is again convex, due to Proposition 5.1 and Proposition 5.5). Hence, according to Lemma 5.4, we can reduce an other player with his nucleolus payoff from the tree. By doing so (in any order), we obtain the reduced tree with respect to a coalition and the nucleolus. Also the Davis and Maschler reduced game can be extended to a reduction of a coalition of players. Hence, although the reductions stated in this subsection are only specified with respect to a single player, Proposition 5.5 remains valid in case one reduces with respect to a coalition of players and their nucleolus payoff.

5.2.4 Standard trees

The definition of a tree with revenues, given in Subsection 5.2.1, is quite general. It allows for certain phenomena which complicate the analysis. Although, we will compute the nucleolus for these trees too, this subsection transforms trees with revenues into, what will be called, standard trees. Furthermore, we discuss the effects of these transformations on the nucleolus. Henceforth, we refer to the nucleolus of a tree with revenues and mean the nucleolus of the corresponding game.

Property (P1). Each edge has strictly positive costs.

In a tree with revenues \(\Gamma\) we have, by definition, non-negative costs. If the cost of edge \(e_p\) is zero, there is no reason to consider \(\pi(p)\) and \(p\) as two different nodes. Therefore, we contract these nodes to one node. Differently, we define the tree with revenues \(\Gamma'\) which is different from \(\Gamma\) only in as far as,

\[
V_0' := V_0 \setminus \{p\} \quad \text{and} \quad N'_{\pi(p)} := N_p \cup N_{\pi(p)}.
\]

![Figure 9. Contraction in a tree with revenues.](image)

It is straightforward to verify that this operation does not effect the TU-game \((N, w_T)\) and thus neither the TU-game \((N, v_T)\). Hence, the nucleolus of \(\Gamma'\) equals the nucleolus of \(\Gamma\).

Property (P2). If a node, not the root, has at most one child, then it is not vacant.

In case a leaf \(p \in V_0\) is vacant, one may delete the edge \(e_p\) as well as node \(p\) from the tree. Because the costs of a coalition equal the costs of a smallest trunk containing the nodes
which inhabit members of the coalition, deletion of a vacant leaf and the connected edge does not effect the TU-game \((N, v_T)\). Hence, this operation does not effect the nucleolus either. Furthermore, a node without inhabitants and one incoming and one outgoing edge does not model anything. Therefore, if \(p \in V_0\) is a node such that there is exactly one node \(q \in V_0\) with \(p = \pi(q)\) and \(N_p = \emptyset\), we delete this node from \(V_0\) and add the costs \(c_p\) to the costs of the edge \(e_q\). To be more formal, we construct a new tree with revenues \(\Gamma'\) which differs from \(\Gamma\) only in

\[ V_0' := V_0 \setminus \{p\} \quad \text{and} \quad c_q' := c_p + c_q. \]

Also this operation does not effect the corresponding TU-game and thus neither the nucleolus.

![Figure 10. Deletion of vacant nodes in a tree with revenues.](image)

Note that deletion of a vacant node is not allowed if it has more than one child.

**Property (P₃).** Each player has a strictly positive revenue.

By definition we have non-negative revenues. If a player has revenue zero, there is no reason for him to be connected to the root, nor is he entitled to attain some part of the total profit. Therefore, one may expect that his payoff in the nucleolus will be zero. This turns out to be true. If \(b_i = 0\) for a certain \(i \in N\), we obtain according to Lemma 5.4 that \(x_i = 0\) for all \(x \in C(v_T)\) and thus in particular for the nucleolus, \(\nu_i = 0\). By reducing the tree with respect to this player \(i\) and his payoff \(\nu_i = 0\), we obtain the reduced tree which almost equals \(\Gamma\), only player \(i\) is missing (see Subsection 5.2.3). According to Proposition 5.5 the nucleolus restricted to \(N \setminus \{i\}\) of the original tree \(\Gamma\) is the nucleolus of the reduced tree (without player \(i\)). Differently, in case one encounters a tree in which some players have revenue zero, one may delete these players from the tree (their payoff is zero in the nucleolus) and calculate the nucleolus for the remaining players. Completing this nucleolus with zeroes, provides us with the nucleolus of the original tree.

**Property (P₄).** The grand coalition is strictly effective, i.e., \(w_\Gamma(N) > w_T(S)\) for all \(S \subseteq N\).

According to this property, the grand coalition can only generate the highest profit by connecting all members of \(N\). For an arbitrary tree this may not be the case. However, if the grand coalition attains the maximal profit by connecting only some of its members, one can expect that these members are not willing to share the profit with the remaining players. So, it seems reasonable that these latter players receive zero in the nucleolus. Indeed, this turns out to be the case. Let \(\Gamma\) be a tree with revenues and let \(S \subseteq N\) be the smallest coalition which yields the highest profit of \(v_T(N)\), i.e., \(w_T(S) = v_T(N)\) and \(w_T(S) > w_T(R)\) whenever \(R \subset S\). Due to the convexity of \((N, w_T)\) this coalition \(S\) is unique. Let \(\nu\) be the nucleolus
of \( \Gamma \). Then \( \nu(N \setminus S) \leq v_T(N) - v_T(S) = 0 \) and thus, by individual rationality, \( \nu_i = 0 \) for all \( i \in N \setminus S \). To obtain a tree in which the largest coalition is strictly effective, we reduce all players in \( N \setminus S \) with their payoff zero from the tree \( \Gamma \). This will result in a tree \( \Gamma' \) in which \( v_T(S) = w_T(S) > w_T(R) \) for all \( R \subseteq S \). Note that the reduced TU-game \( \langle N, v_T \rangle \) is, in fact, the subgame \( \langle S, v_S \rangle \) (i.e., the TU-game of \( \Gamma \) restricted to coalition \( S \)). Again, the reduced game property and Proposition 5.5 yield that the nucleolus of \( \Gamma' \) can be completed (with zeroes) to obtain the nucleolus of the original tree. Hence, we may restrict our analysis for the class of trees with \( w_T(N) > w_T(S) \) for all proper subsets \( S \) of \( N \).

**Remark.** Property (P1) implies in particular that \( b(N(B_p)) - c(B_p) > 0 \) for all branches \( B_p \).

In words, within a branch there is a strictly positive net benefit and therefore it is profitable to connect the inhabitants to the root. Conversely, if all branches have a strictly positive net benefit and Property (P3) holds, then the grand coalition is strictly effective.

The next property is useful from a game theoretical point of view. It gives the zero-normalization of a tree with revenues.

**Property (P3).** The revenue of a player is less than or equal to the total costs of his path to the root.

If in \( \Gamma \) there are certain players \( i \in N \) with \( b_i > c_T(\{i\}) \), we define its zero-normalization \( \Gamma' \) which is different from \( \Gamma \) only in as far as,

\[
\delta_i' := \min\{b_i, c_T(\{i\})\} \quad \text{for all} \quad i \in N.
\]

Observe that if \( \Gamma \) is a tree with revenues which satisfies Property (P1) and Property (P3), these properties will remain for the new tree \( \Gamma' \). Furthermore, the TU-game derived from \( \Gamma' \) is the zero-normalization of the TU-game \( \langle N, v_T \rangle \), as we will demonstrate.

**Claim.** \( v_T(S) = v_T(S) - \sum_{j \in S} v_T(\{j\}) \) for all \( S \subseteq N \).

**Proof.** Let \( I := \{i \in N \mid b_i \geq c_T(\{i\})\} \). Take \( i \in I \) and let \( R \subseteq N \setminus \{i\} \). Then,

\[
w_T(R) + v_T(\{i\}) = b(R) - c_T(R) + b_i - c_T(\{i\}) \\
\leq b(R \cup \{i\}) - c_T(R \cup \{i\}) \\
= w_T(R \cup \{i\}).
\]

Hence, we can conclude that in the tree \( \Gamma \) it is profitable for a coalition to connect all its members who are in \( I \). Differently, for all \( S \subseteq N \) we have that

\[
v_T(S) = \max_{I \cap S \subseteq R \subseteq S} w_T(R).
\]

But since in the tree \( \Gamma' \) we have \( \delta_i' = c_T(\{i\}) \) for the players \( i \in I \), the same statement also holds in the tree \( \Gamma' \). Hence,

\[
v_T(\{i\}) = \max_{I \cap S \subseteq R \subseteq S} w_T(R) \\
= \max_{I \cap S \subseteq R \subseteq S} w_T(R) - \sum_{j \in I \cap S} v_T(\{j\}) = v_T(S) - \sum_{j \in S} v_T(\{j\}).
\]

The last equality follows from the fact that \( v_T(\{j\}) = 0 \) whenever \( j \notin I \).
Hence, the operation on the tree with revenues, as described above, provides us the zero-
normalization of the corresponding TU-game. Since, the nucleolus is covariant (see e.g.,
Peleg and Sudhölter (2003)), the nucleolus \( \nu \) of \( \Gamma \) is equal to \( (\nu'_i + v_T(\{i\}))_{i \in N} \) in which \( \nu' \)
denotes the nucleolus of the zero-normalized tree \( \Gamma' \). Therefore, we restrict our analysis to
trees with revenues in which \( v_T(\{i\}) = 0 \) for all \( i \in N \).

Remark. Property (P3) and Property (P4) imply that there are no players in the root. Indeed,
if there are players \( i \in N_r \) in \( \Gamma \), in the zero-normalized tree \( \Gamma' \) their revenues equal \( b'_i = \min\{b_i, c_T(\{i\})\} = 0 \). Hence, according to Property (P3) their payoff in the nucleolus of \( \Gamma' \)
equals zero. Note that in the nucleolus of the original tree \( \Gamma \) their payoffs will be equal to \( b_i \).
So, in case one encounters a tree in which there are players in the root, one may delete these
players from the tree and give them their revenues as nucleolus payoff.

Property (P6). Every leaf contains at least two players.

Property (P2) tells us that every leaf contains at least one player. In case \( p \in V_0 \) is a leaf
such that \( N_p = \{i\} \) for a certain player \( i \in N \), then we contract this lonely leaf \( p \) with
its predecessor \( \pi(p) \) and decrease the revenue of player \( i \) with \( c_p \). To be more formal, we
construct the tree with revenues \( \Gamma' \) which is different from \( \Gamma \) only in

\[
V' := V \setminus \{p\}, \quad N'_{\pi(p)} := N_{\pi(p)} \cup \{i\} \quad \text{and} \quad b'_i := b_i - c_p .
\]

![Figure 11. Contraction of a lonely leaf.](image)

Observe that \( b'_i := b_i - c_p \) is the net benefit of the branch \( B_p \). So, if \( \Gamma \) satisfies Property (P4),
we have that \( b'_i > 0 \). Hence, within the new tree \( \Gamma' \) we keep strictly positive revenues.
Furthermore, this operation does not effect the corresponding TU-game \( \langle N, v_T \rangle \). Hence, the
nucleolus of \( \Gamma' \) will be equal to the nucleolus of \( \Gamma \).

Property (P7). Only one edge leaves the root.

Let \( \Gamma \) be a tree with revenues in which the root \( r \) has more than one child, i.e., \( |Ch(r)| \geq 2 \),
and let \( \nu \) be its nucleolus. Since, \( \sum_{p \in Ch(r)} v_T(N(B_p)) = v_T(N) \) and \( \{N(B_p)\}_{p \in Ch(r)} \)
is a partition of \( N \), it follows that \( \nu(N(B_p)) = v_T(N) \) for all \( p \in Ch(r) \). Because,
\( v_T(N) = v_T(N(T_p)) \) for all \( p \in Ch(r) \) this yields that

\[
\nu(N(T_p)) = v_T(N(T_p)) \quad \text{for all} \quad p \in Ch(r) .
\]

Take \( p \in Ch(r) \), then by reducing \( \Gamma \) with respect to the players in \( N(T_p) \) and the nucleolus \( \nu \),
we obtain the reduced tree \( \Gamma^-_\nu N(T_p) \) (see Subsection 5.2.3). Since \( \nu(N(T_p)) = v_T(N(T_p)) \)
5.3 The nucleolus of trees with revenues

5.3.1 Excesses at the nucleolus

In this subsection we consider all trees with revenues to be standard (i.e., they satisfy Properties (P₁)–(P₇) mentioned in the previous section). Here, we study the set of coalitions with maximal excess with respect to the nucleolus and we specify the nucleolus payoff for some players. Let us start with the following three lemmas. The proofs are rather technical and are therefore postponed until Section 5.4.

Lemma 5.6. Let \( \nu \) be the nucleolus of \( \Gamma \). If \( S \in D₁(\nu) \), then \( |S| = 1 \) or \( S \) is effective. \( \square \)

The next lemma states that the maximal excess in the nucleolus is strictly less than zero.

Lemma 5.7. If \( \nu \) is the nucleolus of \( \Gamma \), then \( E(\nu) > 0 \). \( \square \)

The next lemma can be derived from the previous one. It states that in the nucleolus no player receives all his revenue and that the inhabitants of a branch together receive strictly less than the net benefit of the branch (i.e., their total revenues minus the total costs of the branch).

Lemma 5.8. If \( \nu \) is the nucleolus of \( \Gamma \), then:

(i) \( 0 < \nu_i < b_i \) for all \( i \in N \),

(ii) \( \nu(N(B_p)) < b(N(B_p)) - c(B_p) \) for all \( p \in V_0 \) with \( \pi(p) \neq r \). \( \square \)

Armed with Lemma 5.6, Lemma 5.7 and Lemma 5.8 one can find those coalitions which determine the nucleolus of a standard tree.

Proposition 5.9. If \( \nu \) is the nucleolus of a standard tree \( \Gamma \), then

\[
D₁(\nu) \subseteq \{ \{i\} \in N : \{N\setminus\{i\}\} \in N, \{N(T_p)\} \subseteq V_0 \}.
\]
Proof. Let $\nu$ be the nucleolus of the standard tree $\Gamma$. Take $S \in D_1(\nu)$ such that $|S| \geq 2$. This implies in particular that $S$ is effective (Lemma 5.6). We distinguish between two cases:

Case (1). If $N(T(S)) \neq N$, then

$$v_T(S) - \nu(S) = b(S) - c(T(S)) - \nu(S) \leq v_T(N(T(S))) - \nu(N(T(S))) = [b(N(T(S))) - \nu(N(T(S)))\setminus S].$$

If $N(T(S)) \neq S$, then Lemma 5.8(i) tells us that $b(N(T(S)))\setminus S > \nu(N(T(S)))\setminus S$. Hence,

$$\text{Exc}(S, \nu) = v_T(S) - \nu(S) < v_T(N(T(S))) - \nu(N(T(S))) = \text{Exc}(N(T(S)), \nu).$$

Because $N(T(S)) \neq N$, this strict inequality contradicts the assumption that $S \in D_1(\nu)$. As a result, $S = N(T(S))$. Next, suppose $T(S)$ is not a maximal trunk. Then there exists a node $p \in Ch(T(S))$ such that $T(S) \subseteq T_p$. Due to Lemma 5.8(ii) we obtain,

$$v_T(N(T(S))) - \nu(N(T(S))) < v_T(N(T(S))) - \nu(N(T(S))) + [b(N(B_p)) - c(B_p) - \nu(N(B_p))].$$

This inequality contradicts the fact that $N(T(S)) \in D_1(\nu)$. Hence, if $S \in D_1(\nu)$ and $N(T(S)) \neq N$, then $S = N(T_p)$ for some node $p \in V_0$.

Case (2). On the other hand, if $N(T(S)) = N$, then we define $U := N \setminus S$. Take $i \in U$ and suppose $S \neq N \setminus \{i\}$. Then we obtain (recall that $c_T(N) = c_T(N \setminus \{i\})$)

$$v_T(N \setminus \{i\}) = v_T(S) \geq b(N \setminus \{i\}) - c_T(N) - [b(S) - c_T(N(T(S)))] = b(N \setminus \{i\}) - b(S) = b(U \setminus \{i\}) > \nu(U \setminus \{i\}) = \nu(N \setminus \{i\}) - \nu(S).$$

The strict inequality follows from Lemma 5.8(i). So, we can conclude that $v_T(S) - \nu(S) < v_T(N \setminus \{i\}) - \nu(N \setminus \{i\})$. But this contradicts again the assumption that $S \in D_1(\nu)$. As a result, $S = N \setminus \{i\}$ for a player $i \in N$. Hence, if $S \in D_1(\nu)$ and $N(T(S)) = N$, then $S = N \setminus \{i\}$ for some player $i \in N$.

According to Theorem 5.3, the set of coalitions $D_1(\nu)$ with maximal excess in $\nu$ contains a partition or an antipartition. Combining this result with Proposition 5.9 yields that an (anti)partition contained in $D_1(\nu)$ is exactly one of the following four types:

(i) $\{N(T_p) : \{i\}_{i\in N(B_p)}\}$ for a certain node $p \in V_0$ with $\pi(p) \neq r$,

(ii) $\{N(T_p) : \{i\}_{i\in N(T_p)}\}$ for a certain trunk $T \subseteq V$,

(iii) $\{N \setminus \{i\}, \{i\}\}$ for some player $i \in N$,

(iv) $\{\{i\}_{i\in N}\}$.  

5.3 The nucleolus of trees with revenues

Remark. Because \( V \) is in particular also a trunk, the antipartitions given in (ii) include the antipartition \( \{ N \setminus \{i\} \}_{i \in N} \). Indeed, if \( T = V \), then \( Ch(T) = \emptyset \) and \( N(T) = N \).

Each of the (anti)partitions stated in the Cases (i)–(iv) will provide us with information on the maximal excess of the nucleolus and mostly even the nucleolus payoff for some players. To make this statement more formal, we first define the following numbers which will turn out to be the basic tool for calculating the nucleolus:

\[
\alpha_p := \frac{b(N(B_p)) - c(B_p)}{|N(B_p)| + 1} \quad \text{for all } p \in V_0 \text{ with } \pi(p) \neq r,
\]
\[
\beta_T := \frac{c(T)}{|N(T)| + |Ch(T)|} \quad \text{for all } T \subseteq V,
\]
\[
\gamma_i := \frac{b_i}{2} \quad \text{for all } i \in N,
\]
\[
\delta := \frac{v_T(N)}{n}.
\]

These numbers are inspired by the numbers defined in Brânzei et al. (2003) for computing the nucleolus of airport profit games. The numbers of type \( \alpha \) compute the net benefit of a branch divided by the number of its inhabitants plus one. The numbers of type \( \beta \) deal with the costs of trunks, i.e., the total costs of a trunk are divided by the number of its inhabitants plus the number of its children. Observe that \( \beta_T = \frac{c(T)}{|N(T)| + |Ch(T)|} \) whenever \( T = V \), since in this case \( N(T) = N \) and \( Ch(T) = \emptyset \). The vector \( \gamma \in \mathbb{R}^N_+ \) corresponds with half of the revenues and the number \( \delta \geq 0 \) equals the total net benefit of the tree divided by the number of all inhabitants. Observe that \( \delta \) is in some sense also of type \( \alpha \). Indeed, since \( V_0 \) is in particular also a branch, \( \delta \) corresponds to \( \alpha_p \) where \( \pi(p) = r \) (and thus \( B_p = V_0 \)), but now one only divides by the number of inhabitants in \( B_p \) (instead of \( |N(B_p)| + 1 \)).

Similar as in setting of Brânzei et al., these numbers will help us to determine which of the Cases (i)–(iv) occurs and more importantly, they will help us to compute the nucleolus payoff for some players. This is stated in Theorem 5.11. The proof of this theorem boils down to the following proposition. A proof for this proposition is given in Section 5.4.

Proposition 5.10. Let \( \Gamma \) be a standard tree and \( \nu \) be its nucleolus. Let \( \mathcal{P} \) be the collection of coalitions:

(i) \( \mathcal{P} := \{N(T_p), \{i\} \in N(B_p)\} \) for a certain node \( p \in V_0 \) with \( \pi(p) \neq r \). Then \( \mathcal{P} \subseteq \mathcal{D}_1(\nu) \) if and only if \( E(\nu) = \alpha_p \) and \( \mathcal{P} \subseteq \mathcal{D}_1(\nu) \) implies \( \nu_i = \alpha_p \) for all \( i \in N(B_p) \).

(ii) \( \mathcal{P} := \{N(T_p)\}_{p \in Ch(T)}, \{N \setminus \{i\}\} \in N(T)\} \) for a certain trunk \( T \subseteq V \). Then \( \mathcal{P} \subseteq \mathcal{D}_1(\nu) \) if and only if \( E(\nu) = \beta_T \) and \( \mathcal{P} \subseteq \mathcal{D}_1(\nu) \) implies \( \nu_i = b_i - \beta_T \) for all \( i \in N(T) \).

(iii) \( \mathcal{P} := \{N \setminus \{i\}, \{i\}\} \) for a certain player \( i \in N \). Then \( \mathcal{P} \subseteq \mathcal{D}_1(\nu) \) if and only if \( E(\nu) = \gamma_i \) and \( \mathcal{P} \subseteq \mathcal{D}_1(\nu) \) implies \( \nu_i = \gamma_i \).

(iv) \( \mathcal{P} := \{i\} \in N \). Then \( \mathcal{P} \subseteq \mathcal{D}_1(\nu) \) if and only if \( E(\nu) = \delta \) and \( \mathcal{P} \subseteq \mathcal{D}_1(\nu) \) implies \( \nu_i = \delta \) for all \( i \in N \)

\( \square \)
Since, the only possible (anti)partitions contained in $D_1(\nu)$ are the collections $P$ given in the Cases (i)–(iv) of the proposition above, we can immediately deduce the following theorem.

**Theorem 5.11.** Let $\Gamma$ be a standard tree and $\nu$ its nucleolus. Then

$$E(\nu) = \min \left\{ \{\alpha_p | p \in V_0, \pi(p) \neq r\}, \{\beta_T \mid \text{trunk } T \subseteq V\}, \{\gamma_i | i \in N\}, \delta \right\}$$

and furthermore:

(A) If $E(\nu) = \alpha_p$ for a certain $p \in V_0$ with $\pi(p) \neq r$, then $\nu_i =\alpha_p$ for all $i \in N(B_p)$,

(B) If $E(\nu) = \beta_T$ for a certain trunk $T \subseteq V$, then $\nu_i = b_i - \beta_T$ for all $i \in N(T)$,

(C) If $E(\nu) = \gamma_i$ for some player $i \in N$, then $\nu_i = \gamma_i$,

(D) If $E(\nu) = \delta$, then $\nu_i = \delta$ for all $i \in N$.

Observe that each of the Cases (A)–(D) in Theorem 5.11 provides us with the nucleolus payoff of at least one player, except in Case (B) when $N(T) = \emptyset$. The remaining part of this subsection tackles this problem.

Let $\nu$ be the nucleolus of $\Gamma$. Assume that there exists a trunk $T \subseteq V$ such that $N(T) = \emptyset$ and $D_1(\nu) = \{N(T_p)\}_{p \in Ch(T)}$. Note that this implies in particular that $|Ch(T)| \geq 2$. Because $N(T) = \emptyset$, the collection $D_1(\nu)$ does not allow us to calculate the payoff in $\nu$ of any of the players. The idea to tackle this problem is to construct a new tree which yields the same nucleolus. This is done as follows. The costs of the edges from the children of trunk $T$ are increased by $\beta_T$ and all costs within the trunk become zero (note that since $N(T) = \emptyset$, we have by definition that $|Ch(T)| \cdot \beta_T = e(T)$). Then, according to Property (P1), we may contract the nodes within the trunk $T$ with the root $r$. Note that $N_r$ remains empty. As a result, we obtain a tree in which the root has several children (i.e., the children of trunk $T$ in the original tree). According to Property (P7) the nucleolus of this tree is the Cartesian product of the nucleoli of the subtrees which are obtained by decomposing to branches $B_p$ rooted in $r$. In addition we prove that this nucleolus is in fact the nucleolus $\nu$ of the original standard tree $\Gamma$. Let us make this idea more formal.

First, we construct the tree $\langle \{V, E, r\}, \{N_p, c'_q\}_{q \in V_0}, (b_i)_{i \in N} \rangle$ in which

$$c'_q := \begin{cases} 0 & \text{if } q \in T, \\ c_q + \beta_T & \text{if } q \in Ch(T), \\ c_q & \text{otherwise.} \end{cases}$$

Then according to Property (P1) we may contract the nodes within the trunk $T$ with the root $r$ which yields the intended tree $\Gamma'$. 

\[\square\]
5.3 The nucleolus of trees with revenues

Figure 12. The construction of $\Gamma'$ out of $\Gamma$. Nodes contained in the trunk $T$ with $N(T) = \emptyset$ are depicted in black and node $p \in Ch(T)$ is one of its children. First, we decrease all costs within $T$ to zero and increase the costs of all children of $T$ by $\beta_T$. Finally, we contract the edges with cost zero.

Lemma 5.12. The nucleoli of $\Gamma$ and of $\Gamma'$ are equal.

Proof. Take $p \in Ch(T)$. We reduce the standard tree $\Gamma$ with respect the players in $N(T_p)$ and $\nu$ (recall that due to Property (P$_2$) we have that $N(T_p) \neq \emptyset$). So, each player $i \in N(T_p)$ makes a payment of $b_i - \nu_i$ to decrease the costs of the path connecting him to the root (see Subsection 5.2.3). Because $D_1(\nu) = \{N(T_q)\}_{q \in Ch(T)}$, Proposition 5.10 yields,

$$E(\nu) = \beta_T = \nu(N(T_p)) - \nu_T(N(T_p)).$$

So, the total payment made by all the inhabitants of the trunk $T_p$, to decrease the costs of the trunk $T_p$, equals,

$$b(N(T_p)) - \nu(N(T_p)) = b(N(T_p)) - \nu_T(N(T_p)) - \beta_T = c(T_p) - \beta_T. \quad (1)$$

The last equality follows from the fact that coalition $N(T_p)$ is effective (Lemma 5.6). Hence, the contribution made by all the players in $N(T_p)$ covers the costs of the trunk $T_p$ minus $\beta_T$.

Next, we prove that the contribution made by the inhabitants of a branch contained in the trunk $T_p$, exceeds the costs of this branch. This means that the costs of all edges in $T_p$ are paid (and therefore become zero), except for the costs of (some) edges on the path from $p$ to the root.

Let $q$ be a node on the path from $p$ to the root (note that $B_q \subseteq T_p$). Lemma 5.8(ii) tells us that

$$b(N(B_q)) - \nu(N(B_q)) > c(B_q). \quad (2)$$

So, according to inequality (2), all the costs of the branches within the trunk $T_p$ are covered by the payments made by the players in $N(T_p)$. Recall from equation (1) that in total $c(T_p) - \beta_T$ is paid. Hence, in the reduced tree with respect to the players in $N(T_p)$ and payoff $\nu$, the costs of all branches within $T_p$ become zero and there remain only some costs on the path from $p$ to the root which in total equals $\beta_T$. Then after contracting all the nodes within $T_p$ with costs zero (Property (P$_1$)) and deleting all the empty nodes on the path from $p$ to the root (Property (P$_2$)), we obtain the tree $\Gamma'$ restricted to the branch $B_p$, in which the costs of the edge $(r, p)$ equals $c_p + \beta_T$. Since, the nucleolus of $\Gamma$ satisfies the reduced game property, the
nucleolus \( \nu \) restricted to the players in \( N(B_p) \) equals the nucleolus of the tree \( \Gamma' \) restricted to this branch \( B_p \). Repeating this argument for each \( p \in Ch(T) \), yields for the nucleoli of \( \Gamma \) and \( \Gamma' \) to coincide. \( \square \)

To conclude, Case (B) in which \( N(T) = \emptyset \) does not allow us to calculate the nucleolus payoff of any of the players. However, we can construct a new tree which yields the same nucleolus (Lemma 5.12). This tree can be decomposed in standard trees and, according to Theorem 5.11, within each of these standard trees at least one of the Cases (A)–(D) occurs again. Property \( (P_7) \) states that the nucleolus is the Cartesian product of the nucleoli of these standard trees.

Knowing how we can solve this problem in Case (B), we have in fact an algorithm for computing the nucleolus of a tree with revenues. The formal description of this algorithm as well as a statement on its complexity are given in the following subsection.

### 5.3.2 Computing the nucleolus

From Theorem 5.11 and the reduced game property, one can derive an algorithm for computing the nucleolus of a tree with revenues. The algorithm can be described as follows (the algorithm does not require a standard tree as input).

We construct a finite sequence of trees, starting with the original tree, where each tree is generated by the preceding one, either via reducing some players, as described in Subsection 5.2.3, or via a transformation, as described above Lemma 5.12. If a tree within this sequence does not satisfy one of the properties \( (P_1) \)–\( (P_7) \), we transform it as described in Subsection 5.2.4 into a standard tree. Let \( \Gamma \) be a standard tree within this sequence of trees. We calculate

\[
\lambda := \min \left\{ \{ \alpha_p \mid p \in V_0, \pi(p) \neq r \} , \{ \beta_T \mid \text{trunk } T \subseteq V \} , \{ \gamma_i \mid i \in N \} , \delta \right\}
\]

and define \( x_i \) according to:

(i) If \( \lambda = \alpha_p \), then \( x_i := \lambda \) for all \( i \in N(B_p) \),

(ii) If \( \lambda = \beta_T \), then \( x_i := b_i - \lambda \) for all \( i \in N(T) \),

(iii) If \( \lambda = \gamma_i \), then \( x_i := \lambda \),

(iv) If \( \lambda = \delta \), then \( x_i := \lambda \) for all \( i \in N \).

Let \( \mathcal{N} \) be the set of players, for which \( x_i \) is defined according to these Conditions (i)–(iv):

- If \( \mathcal{N} = \emptyset \), then we are in Case (B) of Theorem 5.11, with \( N(T) = \emptyset \). Then we transform the tree \( \Gamma \) into a tree \( \Gamma' \), according to the transformation described above Lemma 5.12,

- Otherwise, we reduce the players in \( \mathcal{N} \), as described in Subsection 5.2.3, and obtain a new tree with less players,
5.3 The nucleolus of trees with revenues

- If there are no more players left, the algorithm stops.

**Remark.** Let \( \Gamma \) be a tree with revenues. To obtain a tree satisfying Property (P_{4}) and to compute the numbers \( \alpha_p \) for all \( p \in V_0 \) and to compute \( \delta \), one can proceed as follows. First, delete all players from the tree who’s revenue equals zero. Their nucleolus payoff is zero (Property (P_{3})). Second, compute the net benefit \( b(N_p) - c_p \) for every leaf \( p \). If \( b(N_p) \leq c_p \), then we delete the players in \( N_p \) from the tree, their nucleolus payoff is also zero (Property (P_{4})). Given the strictly positive net benefits of the remaining leaves, the net benefit of a branch \( B_q \), where \( q \) is a predecessor of a leaf, can be computed by adding these net benefits of all its children to \( b(N_q) - c_q \). If this number is less than or equal to zero, we delete all players in the branch \( B_q \). Continuing in this way, one can compute for each branch \( B_p \) its (strictly positive) net benefit and thus also the corresponding \( \alpha_p \). Furthermore, we obtain a tree satisfying Property (P_{4}) and thus we can also compute \( \delta \). Hence, one needs \( O(v) \) calculations (where \( v \) equals the number of nodes in the tree) to compute the \( \alpha_p \)'s and \( \delta \). Clearly, one needs \( O(n) \) calculations for computing the numbers \( \gamma \) (here \( n := |N| \)). To compute the \( \beta_p \)'s in a sophisticated way, we refer to Galil (1980) (see also Maschler et al. (1995)). Both methods need \( O(v \log v) \) calculations.

Note that the algorithm needs at most \( n + v \sim O(n) \) iterations, since in each step at least one player leaves the tree or at least one node is contracted. Hence, the algorithm can be performed in \( O(n \cdot v \log v) \) calculations.

Combining the results of Lemma 5.12 and Theorem 5.11 with the reduced game property of the nucleolus of a tree with revenues (see Subsection 5.2.3), the following result is immediate.

**Corollary 5.13.** The algorithm, as described above, computes the nucleolus of a tree with revenues.

In Subsection 5.3.5 we present a tree with revenues with twelve inhabitants and use the algorithm to compute its nucleolus. In the two forthcoming subsections we study some monotonicity properties satisfied by the nucleolus and we give a characterization for the nucleolus as a solution rule for trees with revenues.

### 5.3.3 Monotonicity properties

This subsection shows that the nucleolus of a (standard) tree with revenues is monotonic in three senses:

(i) If the costs of an edge become larger, none of the players receives more,

(ii) If the revenue of a player increases, none of the players receives less,

(iii) If some of the players are omitted from the nodes, none of the remaining players receives more than before.

To prove these three monotonicity properties of the nucleolus, Theorem 5.11 as well as the reduced game property of the nucleolus of a tree with revenues will play an important role. Let us start by proving that the nucleolus satisfies the *cost-monotonicity* property stated in (i).
Proposition 5.14. Let $\Gamma$ be a tree with revenues. If we increase the costs of a certain edge, the nucleolus payoff weakly decreases.

Proof. By induction on the number of players. Due to efficiency of the nucleolus, the statement holds in case of a tree with one player.

Let $\Gamma$ be a tree with $n$ players. Let $\gamma^* \in V_0$ and increase only the costs of the edge $e_{\gamma^*}$ gradually, i.e., for every $t \geq 0$, define the cost map $c(t) : V \rightarrow \mathbb{R}_+$ by

\[
c_{\gamma^*}(t) := c_{\gamma^*} + t \quad \text{and} \quad c_p(t) := c_p \quad \text{for all nodes } p \in V_0 \text{ with } p \neq \gamma^*.
\]

Given $t \geq 0$, we write $\Gamma(t)$ for the tree with cost function $c(t) : V \rightarrow \mathbb{R}_+$ and we denote $\nu(t) \in R^N$ for its nucleolus. Note that $\Gamma(0) = \Gamma$.

Suppose $\Gamma$ is an example of a tree with revenues such that $\nu_i(0) < \nu_i(t)$ for some $i \in N$ and all $t \in (0, \Delta)$ for some $\Delta > 0$. Let $t \in [0, \Delta)$. After we standardized the tree $\Gamma(t)$, it follows that:

\[
\alpha_p(t) = \begin{cases} 
\alpha_p(0) - \frac{t}{|N(B_p)|+1} & \text{if } p \in B_q, \\
\alpha_p(0) & \text{otherwise},
\end{cases}
\]

\[
\beta_T(t) = \begin{cases} 
\beta_{T}(0) + \frac{t}{|N(T)|+1} & \text{if } \gamma^* \in T, \\
\beta_{T}(0) & \text{otherwise},
\end{cases}
\]

\[
\gamma_i(t) = \gamma_i(0),
\]

\[
\delta(t) = \delta(0) - \frac{t}{n}.
\]

Define for every $t \in [0, \Delta)$:

\[
\lambda^\oplus(t) := \min\{\beta_T(t) \mid \text{trunk } T \subseteq V, \gamma^* \in T\},
\]

\[
\lambda^\ominus(t) := \min\{\{\alpha_p(t) \mid \text{branch } B_p, \gamma^* \notin B_p\} \{\beta_T(t) \mid \text{trunk } T, \gamma^* \notin T\}, \{\gamma_i(t) \mid i \in N\}\},
\]

\[
\lambda^\odot(t) := \min\{\{\alpha_p(t) \mid \text{branch } B_p, \gamma^* \in B_p\}, \delta(t)\}.
\]

Furthermore, for every $t \in [0, \Delta)$ we denote $\lambda(t) := \min\{\lambda^\oplus(t), \lambda^\ominus(t), \lambda^\odot(t)\}$. Observe that $\lambda^\oplus(t)$ is increasing, $\lambda^\ominus(t)$ remains constant and $\lambda^\odot(t)$ is decreasing whenever $t$ increases.

By continuity, we can assume that the argument of $\lambda(0)$ contains the argument of $\lambda(t)$ for all $t \in [0, \Delta)$. We distinguish between the following cases:

- If $\lambda(0) = \lambda^\ominus(0)$, then Theorem 5.11 tells us that for all $t \in [0, \Delta)$,

\[
\nu_j(t) = \lambda^\odot(t) \quad \text{for inhabitants } j \text{ of a certain branch } B_p.
\]

Since, $\lambda^\odot(t)$ is decreasing on the interval $[0, \Delta)$, the nucleolus payoff to the inhabitants of $B_p$ is clearly decreasing. If $B_p = V_0$, then $\nu_j(t) = \delta(t)$ for all players $j \in N$. Hence, in this case the nucleolus payoff is decreasing for all players.
5.3 The nucleolus of trees with revenues

In case \( B_p \neq V_0 \), there are some players left. For these remaining players we can state the following. The contribution of the inhabitants of the branch \( B_p \) exceeds, by definition, the costs of the branch. More precisely, for \( t \in [0, \Delta) \) their contribution equals the costs of \( B_p \) plus \( \alpha_p(t) \). The latter amount is used to diminish the costs of the trunk \( T_p \). Hence, the costs in the reduced tree, with respect to the players in \( B_p \) and their payoff, are increasing on the interval \([0, \Delta)\). Then, by the induction hypothesis and the reduced game property, the nucleolus payoff to the players in \( N(T_p) \) is also decreasing on the interval \([0, \Delta)\).

Both observations contradict the assumption of the existence of a player who’s nucleolus payoff is increasing on \((0, \Delta)\).

- If \( \lambda(0) = \lambda^\oplus(0) < \min\{\lambda^\ominus(0), \lambda^\ominus(0)\} \), then Theorem 5.11 tells us that for all \( t \in [0, \Delta) \),
  \[ \nu_j(t) = b_j - \lambda^\oplus(t) \]
  for inhabitants \( j \) of a certain trunk \( T \).

  Since, \( \lambda^\oplus(t) \) is increasing on the interval \([0, \Delta)\), the nucleolus payoff to the inhabitants of the trunk \( T \) is clearly decreasing, if there are any inhabitants in the trunk. However, for every \( t \in [0, \Delta) \), in the reduced situation, with respect to the players \( j \in N(T) \) and their nucleolus payoff, the total costs of the trunk \( T \) become zero and the costs of all its children increase with \( \lambda^\ominus(t) \) (Lemma 5.12). So, again the costs in the reduced tree are increasing on the interval \([0, \Delta)\). Hence, the induction hypothesis and the reduced game property tells us that the nucleolus payoff for players not in \( N(T) \) is also decreasing on the interval \([0, \Delta)\). Again, this contradicts the assumption of the existence of a player who’s nucleolus payoff is increasing on \((0, \Delta)\).

- Finally, if \( \lambda(0) = \lambda^\ominus(0) < \lambda^\ominus(0) \), then for at least one player, say player \( j \in N \), the nucleolus payoff \( \nu_j(t) \) remains constant for all \( t \in [0, \Delta) \). Again, the induction hypothesis and the reduced game property of the nucleolus, yields that the nucleolus payoff for the remaining players is during the interval \([0, \Delta)\) decreasing. This excludes the existence of a player who’s nucleolus payoff is increasing on \((0, \Delta)\).

Next, we prove that no player receives less under the nucleolus whenever the revenue of a player increases. Differently, the nucleolus of a tree with revenues is revenue-monotonic. The proof of this statement uses a similar reasoning as the previous proof. However, we also use Proposition 5.14.

**Proposition 5.15.** Let \( \Gamma \) be a tree with revenues. If we increase the revenue of a certain player, the nucleolus payoff weakly increases.

**Proof.** By induction on the number of players. Due to efficiency of the nucleolus, the statement holds in case of a tree with one player.

Let \( \Gamma \) be a tree with \( n \) players. Let \( i^* \in N \) and increase the revenue of player \( i^* \) gradually, i.e., for every \( t \geq 0 \), we define \( b_{i^*}(t) := b_{i^*} + t \). Given \( t \geq 0 \), we write \( \Gamma(t) \) for the tree in which player \( i^* \) has revenue \( b_{i^*}(t) = b_{i^*} + t \) and we write \( \nu(t) \in \mathbb{R}_+^N \) for its nucleolus.

Suppose on the contrary that \( \Gamma \) is an example of a tree with revenues such that \( \nu_i(0) > \nu_i(t) \) for some \( i \in N \) and all \( t \in (0, \Delta) \) for some \( \Delta > 0 \).
Let \( t \in [0, \Delta) \). After we standardized the tree \( \Gamma(t) \), it follows that:
\[
\alpha_p(t) = \begin{cases} 
\alpha_p(0) + \frac{t}{|N(B_p)| + 1} & \text{if } i^* \in N(B_p), \\
\alpha_q(0) & \text{otherwise},
\end{cases}
\]
\[
\beta_T(t) = \beta_T(0) \text{ for all trunks } T \subseteq V,
\]
\[
\gamma_j(t) = \begin{cases} 
\gamma_j(0) + \frac{t}{2} & \text{if } j = i^*, \\
\gamma_j(0) & \text{otherwise},
\end{cases}
\]
\[
\delta(t) = \delta(0) + \frac{t}{n}.
\]

Define for every \( t \in [0, \Delta) \),
\[
\lambda(t) = \min \{ \{ \alpha_p(t) | p \in V_0, \pi(p) \neq r \}, \{ \beta_T(t) | \text{trunk } T \subseteq V \}, \{ \gamma_j(t) | i \in N \}, \delta(t) \}. 
\]

By continuity, we can assume that the argument of \( \lambda(0) \) contains the argument of \( \lambda(t) \) for all \( t \in [0, \Delta) \). Similarly, as in the proof of Proposition 5.14, one needs to distinguish between several cases. However, the result of Proposition 5.14 is also used.

- Assume that the minimum \( \lambda(0) = \alpha_p(0) \) for a certain node \( p \in V_0, \pi(p) \neq r \) and this minimum is not attained at a number of type \( \beta, \gamma \) or \( \delta \). Then, due to Theorem 5.11, we have for all \( t \in [0, \Delta) \)
\[
\nu_j(t) = \alpha_p(t) \text{ for all } j \in N(B_p).
\]

Since, \( \alpha_p(t) \) is (weakly) increasing on the interval \([0, \Delta)\), the nucleolus payoff to the inhabitants of the branch \( B_p \) is also increasing on this interval. Using a similar reasoning as in the proof of Proposition 5.14, it follows that the costs in the reduced tree, with respect to the players in \( N(B_p) \) and their nucleolus payoff, are decreasing. Hence, Proposition 5.14, the induction hypothesis and the reduced game property of the nucleolus tell us that the nucleolus payoff for all remaining players is also increasing. This contradicts the existence of a player who’s nucleolus payoff is strictly decreasing.

- In a similar way one can obtain the desired contradiction whenever \( \lambda(0) \) equals \( \beta_T(0), \gamma_j(0) \) or \( \delta(0) \). The proofs are left to the reader. \( \square \)

In the next monotonicity property of the nucleolus Proposition 5.14 plays also an important role. This property states that the nucleolus is population-monotonic.

**Proposition 5.16.** Let \( \Gamma \) be a tree with revenues. If we decrease the number of inhabitants in a certain node, the nucleolus payoff of any remaining player weakly decreases.

**Proof.** Let \( \Gamma \) be a tree with revenues. Let \( p \in V_0 \) and take \( i \in N_p \). We denote \( \Gamma^{-i} \) for the tree with revenues in which player \( i \) is left out. Note that the costs in the reduced tree \( \Gamma^{-i} \), with respect to player \( i \) and the nucleolus payoff \( \nu_i(\Gamma) \), are less than or equal to the costs in the tree \( \Gamma^{-i} \). This follows from the fact that \( b_i - \nu_i(\Gamma) \geq 0 \) (Lemma 5.4) and thus the costs of the path \( T(i) \) in the reduced tree have decreased. Hence, Proposition 5.14 tells us that for all \( j \in N \setminus \{i\} \) we have
5.3 The nucleolus of trees with revenues

\[ \nu_j(\Gamma_{-i}) \geq \nu_j(\Gamma). \]

Using the reduced game property, the inequality above yields that \( \nu_j(\Gamma) \geq \nu_j(\Gamma_{-i}) \) for all \( j \in N \setminus \{i\} \). Hence, the nucleolus payoff of the remaining players has weakly decreased.

A direct consequence of Proposition 5.16 is that the nucleoli of all subgames of a tree with revenues form a population monotonic allocation scheme (pmas) (Thomson (1983), Sprumont (1990)). A pmas of a TU-game \( (N,v) \) gives a core element for every subgame \( (S,v_S) \) such that every player \( i \in N \) gets a weakly higher payoff in larger coalitions. To be more formal, a pmas is a table \( \{x_{Si}\}_{S \subseteq N, i \in N} \) with the following properties:

1. \( x_{Si} = 0 \) for all \( S \subseteq N \) whenever \( i \notin S \),
2. \( \sum_{i \in S} x_{Si} = v(S) \) for all \( S \subseteq N \),
3. \( x_{Ri} \leq x_{Si} \) for all \( i \in N \) whenever \( R \subseteq S \).

The result of Proposition 5.16 implies that the nucleolus of a tree with revenues can be extended into a pmas, in the following way.

**Corollary 5.17.** The nucleoli of all subgames of a tree with revenues form a population monotonic allocation scheme.

### 5.3.4 A characterization

This section is devoted to provide a characterization for the nucleolus as a solution rule for a tree with revenues. A single valued solution rule \( \tau \) assigns a vector \( \tau(\Gamma) \in \mathbb{R}^N \) to each tree with revenues \( \Gamma \). The number \( \tau_i(\Gamma) \) denotes the payoff to player \( i \in N \). Let us start by providing some desirable properties for an (arbitrary) solution rule for trees with revenues.

**Reasonableness.** A solution rule \( \tau \) is reasonable if

\[ 0 \leq b_i - \tau_i(\Gamma) \leq c(T(\{i\})) \quad \text{for all } i \in N. \]

The difference \( b_i - \tau_i(\Gamma) \) can be understood as player’s contribution to the total costs of the tree. In Subsection 5.2.3 we already discussed that it is reasonable in a non-technical meaning of the word for this difference to be positive as well as less than total costs \( c(T(\{i\})) \) of the path \( T(\{i\}) \), connecting player \( i \) with the root. The name reasonableness (on both sides) for this property is due to Sudhölter (1997).

**Consistency.** A solution rule \( \tau \) is consistent if for every tree with revenues \( \Gamma \) and for each player \( i \in N \) with \( z := \tau_i(\Gamma) \) satisfying \( 0 \leq b_i - z \leq c(T(\{i\})) \) it holds that \( \tau_j(\Gamma_{-i}) = \tau_j(\Gamma) \) for all \( j \in N \setminus \{i\} \).

This notion of consistency is what Potters and Sudhölter (1999) call \( \nu \)-consistency. It means that whenever a player \( i \in N \) receives a payoff \( \tau_i(\Gamma) \) which is reasonable and the (non-negative) difference \( b_i - \tau_i(\Gamma) \) is used to diminish the costs of the tree as described in Subsection 5.2.3, the payoff according to the solution rule to the remaining players in the reduced tree equals the payoff in the original tree.
Cost-monotonicity. A solution rule $\tau$ is cost-monotonic if
\[ \tau_i(\Gamma') \leq \tau_i(\Gamma) \text{ for all } i \in N \text{ whenever } c'_q \geq c_q \text{ for all nodes } q \in V_0. \]

Cost-monotonicity states that no player receives a higher payoff whenever the costs of the edges are increased.

Our first result in this subsection states that if two solution rules assign equal solutions for trees with less than three players and if they both satisfy reasonableness, consistency and cost-monotonicity, then they coincide for every tree with revenues.

Proposition 5.18. If $\tau$ and $\sigma$ are two solution rules both satisfying reasonableness, consistency and cost-monotonicity and moreover they coincide whenever $|N| = 2$, then $\tau = \sigma$.

Proof. Let $\tau$ and $\sigma$ be two solution rules satisfying the conditions mentioned in the proposition and suppose $\tau \neq \sigma$. Let $\Gamma$ be a tree with revenues, with a minimal number of players such that $\tau(\Gamma) \neq \sigma(\Gamma)$.

If there is a player $i \in N$ such that $\tau_i(\Gamma) > \sigma_i(\Gamma)$, then, due to reasonableness, we can reduce this player with either the payoff $\tau_i(\Gamma)$ or $\sigma_i(\Gamma)$ from the tree. Note that $b_i - \tau_i(\Gamma) < b_i - \sigma_i(\Gamma)$ and thus the costs in the reduced tree $\Gamma'_{\tau_i(\Gamma)}$ are larger than the costs in the reduced tree $\Gamma'_{\sigma_i(\Gamma)}$. Hence, by cost-monotonicity, we have that
\[ \tau(\Gamma'_{\tau_i(\Gamma)}) \leq \tau(\Gamma'_{\sigma_i(\Gamma)}) \quad \text{and} \quad \sigma(\Gamma'_{\tau_i(\Gamma)}) \leq \sigma(\Gamma'_{\sigma_i(\Gamma)}). \]

Recall that $\Gamma$ is a tree with the smallest number of players for which $\tau$ and $\sigma$ differ. Therefore, by consistency it follows that for all $k \neq i$,
\[ \tau_k(\Gamma) = \tau_k(\Gamma'_{\tau_i(\Gamma)}) = \sigma_k(\Gamma'_{\tau_i(\Gamma)}) \leq \sigma_k(\Gamma'_{\sigma_i(\Gamma)}) = \sigma_k(\Gamma). \]

If there exists a player $j \neq i$ with $\tau_j(\Gamma) < \sigma_j(\Gamma)$, one can prove in exactly the same manner that for all $k \neq j$,
\[ \tau_k(\Gamma) \geq \sigma_k(\Gamma). \]

Hence, there exists a player $k \in N$ with $\tau_k(\Gamma) = \sigma_k(\Gamma)$. By reducing this player and use again the fact that $\Gamma$ is a tree with the smallest number of players for which $\tau$ and $\sigma$ differ, we obtain, again by consistency, that $\tau_{\ell}(\Gamma) = \sigma_{\ell}(\Gamma)$ for all $\ell \neq k$. Hence, $\tau(\Gamma) = \sigma(\Gamma)$. $\square$

Due to Proposition 5.18, we may restrict our attention to trees with two inhabitants. Although, this observation seems to simplify the analysis quite a lot, it is far from evident to characterize the nucleolus of a tree with two players. This is mainly due to the fact that the combinatorial structure of a tree with two players may still be quite complicated. To achieve our goal of characterizing the nucleolus of a tree with revenues, we introduce four additional properties of a solution rule.

Profit-making. A solution rule $\tau$ satisfies profit-making whenever $b(N_p) \leq c_p$ for a leaf $p \in V_0$ implies
\[ \tau_i(\Gamma) = 0 \text{ for all } i \in N_p \text{ and } \tau_i(\Gamma) = \tau_i(\Gamma') \text{ for all } i \notin N_p, \]
in which $\Gamma'$ differs from $\Gamma$ only as far as, $V' := V \setminus \{p\}$ and $N' := N \setminus N_p$. $\circ$
Profit-making states that inhabitants of a leaf generating a non-positive net benefit receive zero as payoff. If in a leaf there is no strictly positive net benefit, there is no reason for the inhabitants to be connected to the root. Therefore, it seems reasonable that these players will refrain from contributing to the costs and thus they cannot earn their revenues. On the other hand, it is equally reasonable for the remaining players not to share their total profit with these players. Therefore, the remaining players are indifferent whether this leaf is present or not.

**Minimal obligation.** A solution rule $\tau$ satisfies minimal obligation whenever $N_p = \{i\}$ for a leaf $p \in V_0$ and $b_i > c_p$ implies $\tau_i(\Gamma) = \tau_i(\Gamma')$ in which the tree with revenues $\Gamma'$ is different from $\Gamma$ only in as far as, $N_p' := \emptyset, b_i' := b_i - c_p$ and $N_{\pi(p)}' := N_{\pi(p)} \cup \{i\}$. Minimal obligation states that whenever a leaf contains a single player, this player may move to its predecessor if he has paid the costs of the edge connecting him to his predecessor. Differently, minimal obligation states that if such a ‘lonely’ player wants to receive some payoff, he at least has to pay his marginal costs.

**Minimal rights first.** A solution rule $\tau$ satisfies minimal rights first whenever $b_i > c(T(\{i\}))$ for a certain player $i \in N$ implies

$$\tau_i(\Gamma) = \tau_i(\Gamma') + [b_i - c(T(\{i\}))]$$

in which the tree with revenues $\Gamma'$ is different from $\Gamma$ only in as far as, $b_i' := c(T(\{i\}))$. Minimal rights first states that whenever a player may earn more (from being connected to the root) than his stand-alone costs (the total costs of the path to the root), he may receive already his (individual) profit and participate in the tree as if he has an (individual) profit of zero.

**Indispensability.** A solution rule $\tau$ satisfies indispensability whenever the existence of an inhabitant $i \in N_p$ of a leaf $p \in V_0$ with $b_i \geq b(N_p) - c_p$, implies $\tau_i(\Gamma) \geq \tau_j(\Gamma)$ for all $j \in N_p$. If the revenue of an inhabitant of a leaf exceeds the net benefit of this leaf, this player can be seen as an indispensable player, since without him the net benefit would be less than or equal to zero. Indispensability states that such a player should receive at least as much payoff as any other inhabitant of the leaf.

Next, we prove that if a solution rule $\tau$ satisfies consistency, cost-monotonicity, profit-making and minimal rights first, then $\tau$ satisfies one part of reasonableness, namely, $b_i - \tau_i(\Gamma) \leq c(T(\{i\}))$ for all $i \in N$. Therefore, we weaken the property of reasonableness, to reasonableness on one side (i.e., each player has a non-negative contribution to the costs).

**Reasonableness on one side.** A solution rule $\tau$ is reasonable on one side if

$$b_i - \tau_i(\Gamma) \geq 0 \text{ for all } i \in N.$$
Lemma 5.19. Let $\tau$ be a solution rule which satisfies consistency, cost-monotonicity, profit-making, minimal rights first and reasonableness on one side, then $\tau$ is reasonable (on both sides).

Proof. Let $\tau$ be a solution rule, satisfying the five properties mentioned in the lemma. We first prove, using consistency, cost-monotonicity and profit-making, that $\tau_i(\Gamma) \geq 0$ for all $i \in N$ and for every tree with revenues $\Gamma$.

Let $p \in V_0$ be a leaf of $\Gamma$ such that $b(N_p) > c_p$. Let $\Gamma'$ be the tree in which only the costs of the edge $c_p$ has increased by $b(N_p) - c_p$. Then, due to profit-making and cost-monotonicity, it follows that $\tau_i(\Gamma) \geq \tau_i(\Gamma') = 0$ for all $i \in N_p$. Furthermore,

$$0 \leq b_i - \tau_i(\Gamma') = b_i \leq b(N_p) \leq c'(T(\{i\}))$$

for all $i \in N_p$.

Thus, we can reduce the inhabitants of $N_p$ from the tree $\Gamma'$ with their payoff zero. Hence, consistency and again cost-monotonicity tells us that

$$\tau_j(\Gamma) \geq \tau_j(\Gamma') = \tau_j((\Gamma')_{0}^{N_p})$$

for all players $j \notin N_p$.

In the reduced tree $(\Gamma')_{0}^{N_p}$ we can take an other leaf $q$, with a strictly positive net benefit, and increase again only the costs on the edge $e_q$ until $q$ generates a net profit of zero. Using again profit-making and cost-monotonicity, it follows that $\tau_j(\Gamma) \geq \tau_j(\Gamma') \geq 0$ for all $j \in N_q$. Continuing in this way, one can derive that $\tau_i(\Gamma) \geq 0$ for all $i \in N$.

Now, we are able to prove that

$$b_i - \tau_i(\Gamma) \leq c(T(\{i\}))$$

for all $i \in N$.

Take $i \in N$. If $b_i \leq c(T(\{i\}))$, then $b_i - \tau_i(\Gamma) \leq c(T(\{i\}))$ since $\tau_i(\Gamma) \geq 0$. On the other hand, if $b_i \geq c(T(\{i\}))$, then due to minimal rights first it follows that

$$b_i - \tau_i(\Gamma) = b_i - (\tau_i(\Gamma') + b_i - c(T(\{i\}))) = c(T(\{i\})) - \tau_i(\Gamma').$$

Here, $\Gamma'$ differs from $\Gamma$ only as far as, the revenue of player $i$ equals $b_i' := c(T(\{i\}))$. Hence, again since $\tau_i(\Gamma') \geq 0$ it follows that $b_i - \tau_i(\Gamma) \leq c(T(\{i\}))$.

The main result of this section is that the properties stated in this section characterize the nucleolus of a tree with revenues.

Theorem 5.20. The nucleolus is the only solution rule for trees with revenues that satisfies, reasonableness on one side, consistency, cost-monotonicity, profit-making, minimal obligation, minimal rights first and indispensability.

Proof. Existence-part) Since, on the level of TU-games, consistency is the reduced game property, the nucleolus satisfies consistency (see Subsection 5.2.3). The nucleolus of a tree with revenues also satisfies profit-making and reasonableness on one side, since it is a core element of the corresponding TU-game (see Subsection 5.2.4 and Lemma 5.4, respectively). Proposition 5.14 tells us that the nucleolus satisfies cost-monotonicity. Because the transformation on the tree described in minimal obligation does not affect the corresponding TU-game and since the transformation on the tree described in minimal rights first provides the
5.3 The nucleolus of trees with revenues

zero-normalization of the corresponding TU-game, also these two properties are satisfied by the nucleolus (see again Subsection 5.2.4). To prove that the nucleolus also satisfies indispen-
sability, we need to do some work.

Let \( p \in V_0 \) be a leaf with \( b(N_p) \geq c_p \). We reduce all players in \( N \setminus N_p \) with their nucleolus payoff from the tree. Note that the players in the reduced tree

\[
\Gamma' := \Gamma_{\nu\Gamma}^{-N \setminus N_p}
\]

are inhabitants of node \( p \in V_0 \). Furthermore, in this reduced tree, the costs of the path from this node \( p \) to the root \( \tau \) equal \( c_p \) plus some costs which have not been covered by the players in \( N \setminus N_p \) who left the tree. Define \( U \subseteq N_p \) by

\[
i \in U \iff b_i \geq b(N_p) - c_p.
\]

Then \( b(N_p \setminus \{i\}) - c_p \leq 0 \) for all \( i \in U \). Hence, in the (Davis and Maschler) reduced game \( (N_p, \nu_{\Gamma'}) \) corresponding to the reduced tree \( \Gamma' \) it holds that \( \nu_{\Gamma'}(S) = 0 \) whenever \( U \subseteq S \). It is not difficult to verify that for all \( i \in U \) it holds that \( \nu_i(\Gamma') \geq \nu_i(\Gamma') \) whenever \( j \not\in U \).

Furthermore, using a symmetry argument (players in \( U \) are interchangeable), it follows that \( \nu_i(\Gamma') = \nu_j(\Gamma') \) for all \( i, j \in U \). Using the reduced game property of the nucleolus of a tree with revenues, it follows that the nucleolus satisfies indispen-
sability. This completes the ‘Existence-part’ of the proof.

Uniqueness-part) Let \( \tau \) be a solution rule for tree with revenues, satisfying the seven properties mentioned in the theorem. Let us first investigate how this solution rule \( \tau \) behaves on the following three simple trees with revenues:

\[
\Gamma_{Single}(b, c) \quad \Gamma_{Fork}(b_1, b_2) \quad \Gamma_{Airport}(b_1, b_2)
\]

Figure 13. Three simple trees with revenues. The root is depicted in black.

Let us start by analyzing the left tree in Figure 13, denoted by \( \Gamma_{Single}(b, c) \). If \( b \leq c \), then due
to profit-making, it follows that \( \tau(\Gamma_{Single}(b, c)) = 0 \). If \( b > c \), then due to minimal rights
first, it follows that \( \tau(\Gamma_{Single}(b, c)) = \tau_1(\Gamma_{Single}(c, c)) + [b - c] = b - c \). The last equality follows again from profit-making. Hence, as a result we obtain, \( \tau(\Gamma_{Single}(b, c)) = [b - c] \).

To analyze the middle tree in Figure 13, denoted by \( \Gamma_{Fork}(b_1, b_2) \), we can proceed as in the
previous tree. If \( b_1 \leq c_1 \), then due to profit-making it follows that \( \tau_1(\Gamma_{Fork}(b_1, b_2)) = 0 \). On the other hand, if \( b_1 > c_1 \), then minimal rights first states that \( \tau_1(\Gamma_{Fork}(b_1, b_2)) = \tau_1(\Gamma_{Fork}(c_1, b_2)) + [b_1 - c_1] = b_1 - c_1 \). Hence, \( \tau_1(\Gamma_{Fork}(b_1, b_2)) = [b_1 - c_1] + \). A similar statement can be proved for player 2.

If in the right tree in Figure 13, denoted by \( \Gamma_{Airport}(b_1, b_2) \), it holds that \( b_1 + b_2 \leq c \), then due
to profit-making it follows that \( \tau_1(\Gamma_{Airport}(b_1, b_2)) = \tau_2(\Gamma_{Airport}(b_1, b_2)) = 0 \). So, we can assume that \( b_1 + b_2 > c \). If \( b_1 \geq c \) or \( b_2 \geq c \), we first apply minimal rights first. Thus, we can
The Nucleolus of Trees with Revenues

assume that $b_1 \leq c$ and $b_2 \leq c$. This means that $b_1 \geq b_1 + b_2 - c$ and $b_2 \geq b_1 + b_2 - c$. Hence, both players are indispensable which results in $\tau_1(\Gamma_{\text{Airport}}(b_1, b_2)) = \tau_2(\Gamma_{\text{Airport}}(b_1, b_2))$. Denote $z := \tau_2(\Gamma_{\text{Airport}}(b_1, b_2))$. Due to Lemma 5.19, $\tau$ satisfies reasonableness on both sides. So, we can reduce player 2 from the tree with his payoff $z$. By doing so, we obtain the reduced tree, $\Gamma_{\text{single}}(b_1, c - (b_2 - z))$. Hence, consistency combined with our previous results, tells us that $z = [b_1 - (c - (b_2 - z))]_+ = [b_1 + b_2 - c - z]_+$, and thus $z = \frac{1}{2}(b_1 + b_2 - c)$.

So far, we can conclude that for the two trees in Figure 13 with two players the payoff according to the solution rule $\tau$ equals for $i = 1, 2$,

$$[b_i - c(T(\{i\}))_+ + \frac{1}{2} \left([b_1 + b_2 - c(T(\{1, 2\}))_+ - \sum_{j=1}^{2} [b_j - c(T(\{j\}))_+\right].$$

This coincides with the standard solution (Aumann and Maschler (1985)) of the corresponding TU-game and thus $\tau$ coincides with the nucleolus for the trees in Figure 13, with two players.

Consider now an arbitrary tree with two players. Due to profit-making, we may assume that there are no branches without any inhabitants. If the two players use at least one common edge, to be connected to the root, then after using the properties of profit-making and minimal obligation of $\tau$ recursively, one can transform this tree into the right tree in Figure 13. If the paths from both players are disjoint, one can proceed in a similar way to transform the tree into the middle tree in Figure 13. As a result, we can conclude that $\tau$ equals the nucleolus for a tree with two inhabitants.

Due to Lemma 5.19, $\tau$ satisfies reasonableness on both sides and thus, according to Proposition 5.18, it follows that this solution rule $\tau$ coincides with the nucleolus, since it coincides with the nucleolus for trees with two players. This completes the ‘uniqueness-part’ and the proof.

We end this section, by discussing the logic independence of the properties used in our characterization for the nucleolus. Unfortunately, we were not able for each property to prove its logical independence from the remaining ones, nor to prove a dependence relation. However, for two of the properties, mentioned in Theorem 5.20, we have been able to devise examples which illustrate their logic independence from the other six properties. The examples are listed below.

The first example shows the logic independence of consistency with respect to the other properties mentioned in Theorem 5.20.

**Example (Consistency).** Consider the solution rule $\tau$ which assigns to each tree with revenues $\Gamma$ the following payoff to player $i \in N$,

$$\tau_i(\Gamma) := [b_i - c(T(\{i\}))].$$

Then $\tau$ clearly satisfies reasonableness on one side and cost-monotonicity. If $\Gamma$ is a tree with revenues in which there exists a leaf $p \in V_0$ such that $b(N_p) \leq c_p$, then $b_i \leq c_p \leq c(T(\{i\}))$ for all $i \in N_p$. Hence, $\tau_i(\Gamma) = 0$ for all $i \in N_p$. Furthermore, by definition it follows that $\tau_j(\Gamma') = \tau_j(\Gamma')$ for all $j \notin N_p$, in which $V' := V \setminus \{p\}$. Hence, $\tau$ satisfies profit-making. Note
that minimal obligation and minimal rights first are also satisfied by $\tau$. Next, we show that $\tau$ also satisfies indispensability. Let $p \in V_0$ be a leaf such that $b(N_p) \geq c_p$ and $b_i \geq b(N_p) - c_p$ for a certain player $i \in N_p$. Take $j \in N_p$ with $j \neq i$. Then $b_i \geq b(N_p) - c_p \geq b_i + b_j - c_p$. Hence, $b_j \leq c_p \leq c(T(\{j\}))$ and thus $\tau_j(\Gamma) = 0$. Hence, $\tau_i(\Gamma) \geq \tau_j(\Gamma)$ for all inhabitants $j \in N_p$. Finally, since $\tau$ does not coincide with the nucleolus, we can conclude that it does not satisfy consistency.

Also minimal rights first cannot be proven from the other six properties.

**Example (Minimal rights first).** Consider the solution rule $\tau$ which assigns to each tree with revenues $\Gamma$ the following payoff to player $i \in N$,

$$\tau_i(\Gamma) := 0.$$ 

Then it is straightforward that this solution rule satisfies reasonableness on one side, consistency, cost-monotonicity, profit-making, minimal obligation and indispensability. However, it clearly does not satisfy minimal rights first.

### 5.3.5 Example

In this subsection we present an example of a tree with revenues and use the algorithm, given in Subsection 5.3.2, to compute its nucleolus.

**Example.** Consider the tree with revenues $\Gamma$ with 12 players as depicted in Figure 14. The revenues of the players are written between brackets and the costs are given next to the edges. There are three vacant nodes, including the root.

![Figure 14](image-url)

**Figure 14.** The tree with revenues $\Gamma$. The three black nodes, including the root, are vacant. Node $s$ and node $v$ have exactly one inhabitant and the remaining nodes two. The revenues of the players are written between brackets.

The nucleolus of $\Gamma$ equals:

<table>
<thead>
<tr>
<th>Player $i$ in node</th>
<th>$s$</th>
<th>$t$</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
<th>$p$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue $b_i$</td>
<td>8</td>
<td>(32,27)</td>
<td>(31,34)</td>
<td>28</td>
<td>(30,40)</td>
<td>(24,30)</td>
<td>(20,34)</td>
</tr>
<tr>
<td>Nucleolus $\nu_i$</td>
<td>4</td>
<td>(13,13)</td>
<td>(16,19)</td>
<td>17</td>
<td>(18,28)</td>
<td>(10,10)</td>
<td>(10,16)</td>
</tr>
<tr>
<td>Payment $b_i - \nu_i$</td>
<td>4</td>
<td>(19,14)</td>
<td>(15,15)</td>
<td>11</td>
<td>(12,12)</td>
<td>(14,20)</td>
<td>(10,18)</td>
</tr>
</tbody>
</table>
Observe that the payments on the total costs of the tree by players in the same node can be different. Furthermore, although \( v < q \), there is an inhabitant of node \( q \), who contributes 10 to the costs of the tree which is strictly less than the payment made by the player in node \( v \).

Let us now use the algorithm of Subsection 5.3.2 to compute the nucleolus, stated above.

**Step 1.** The tree in Figure 14 is standard. Among the numbers \( \alpha \), the minimum is taken in \( \alpha_p = 13 \cdot (54 - 24) = 10 \). Among the numbers \( \beta \), the minimum is taken at the trunk \( T \) containing the tree black nodes and node \( s \), yielding \( \beta_T = 13 \cdot (24 + 7 + 5) = 7 \frac{7}{1} \). The minimum among the \( \gamma \)'s is clearly attained by the player \( j \) in node \( s \), yielding \( \gamma_j = 4 \). Finally, \( \delta = 13 \cdot (338 - 164) = 14 \frac{1}{2} \). Hence, \( \nu_j = \gamma_j = 4 \). Reducing the tree with respect to this player \( j \in N_s \) and his payoff \( \nu_j = 4 \), yields the following standard tree.

![Figure 14(i)](image)

**Figure 14(i).** The reduced tree with revenues. The nucleolus payoff, so far, equals:

<table>
<thead>
<tr>
<th>Player in node</th>
<th>s</th>
<th>t</th>
<th>u</th>
<th>v</th>
<th>w</th>
<th>p</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nucleolus</td>
<td>4</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

**Step 2.** The numbers \( \alpha \) remain the same and thus their minimum equals 10. The minimum among the \( \gamma \)'s has increased to \( \frac{20}{7} = 10 \). Also \( \delta \) has increased. However, the \( \beta \)'s have changed and the minimum is taken in the trunk \( T \) containing the four black nodes, yielding \( \beta_T = 13 \cdot (20 + 7 + 5) = 8 \). Since, \( N(T) = \emptyset \), we obtain the following tree.

![Figure 14(ii)](image)

**Figure 14(ii).** Increasing the costs of the children with 8 and contracting the vacant nodes.

**Step 3.** The tree in Figure 14(ii) is not standard. First, we decompose this tree into the rooted branches \( B_t, B_u, B_v \) and \( B_w \) (Property (P)). The tree restricted to the branch \( B_t \) is standard and the minimum occurs at \( \delta = 13 \). Hence, the nucleolus payoff for both inhabitants of node \( t \) equals 13. The tree restricted to \( B_u \) is not standard, since it does not satisfy Property (P). After zero-normalization (i.e., the revenues become (30, 30) and the inhabitants of node \( u \) already receive \( (31 - 30, 34 - 30) \) in the nucleolus), the minimum is attained at \( \delta = 15 \). So, the nucleolus of the zero-normalized tree is \( (15, 15) \). Hence, the nucleolus payoff for the players in \( u \) is \( 1 + 15 = 16 \) and \( 4 + 15 = 19 \), respectively. Similarly, one can compute the
nucleolus for the tree restricted to $B_w$ which equals $(6 + 12, 16 + 12) = (18, 28)$. This leaves us with the tree restricted to $B_v$. Here, the minimum is attained at $\alpha_p = 10$, yielding for the players in node $p$ to receive $(10, 10)$ in the nucleolus. Hence, by reducing all these players with their payoff from the tree, we obtain the following reduced tree and its corresponding standard tree.

Figure 14(iii). The reduction with respect to the players in the nodes $t, u, p$ and $w$ and the zero-normalization. So far, the nucleolus equals (the player in node $v$ already receives $28 - 22 = 6$ and is therefore marked):

<table>
<thead>
<tr>
<th>Player in node</th>
<th>4</th>
<th>(13,13)</th>
<th>(16,19)</th>
<th>6*</th>
<th>(18,28)</th>
<th>(10,10)</th>
<th>(−,−)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nucleolus</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Step 4. The minimum in the standard tree of Figure 14(iii) is taken in $\gamma_j = 10$ for the player $j$ in node $q$ with $b_j = 20$. This yields a payoff in the nucleolus for this player to be 10. Reducing this player from the standard tree in Figure 14(iii) yields the following tree.

Figure 14(iv). The reduction with respect to the player in node $q$ and its standard tree. So far, the nucleolus equals (the player in node $q$ already receives $34 - (7 + 22) = 5$ and is therefore marked):

<table>
<thead>
<tr>
<th>Player in node</th>
<th>4</th>
<th>(13,13)</th>
<th>(16,19)</th>
<th>6*</th>
<th>(18,28)</th>
<th>(10,10)</th>
<th>(10,5*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nucleolus</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Step 5. Finally, the minimum within the standard tree in Figure 14(iv), is taken in $\delta = 11$. This yields for this tree, the nucleolus payoff for both players to be 11. Hence, the inhabitant of node $v$ of the original tree receives $6 + 11 = 17$ as nucleolus payoff and the player in node $q$, who already received 5, receives now in total $5 + 11 = 16$ as nucleolus payoff.

5.4 Proofs

This section provides a proof for Proposition 5.5, for the Lemmas 5.6, 5.7 and 5.8 and for Proposition 5.10.

Proof of Proposition 5.5. Take $i \in N$, say $i \in N_p$, for a certain node $p \in V_0$ and let $x \in C(v_T)$.

Claim. $c_{T^-}(S) = \min \{c_{T}(S \cup \{i\}) - (b_i - x_i), c_{T}(S)\}$ for all $S \subseteq N \setminus \{i\}$.

Proof. Take $S \subseteq N \setminus \{i\}$, we distinguish between two cases. In case $p \in T(S)$, then $T(S \cup \{i\}) = T(S)$ and furthermore,
Lemma 5.4), this yields

\[ c_{T^2}(S) = c(T(S)) - (b_i - x_i) \]
\[ = c(T(S \cup \{i\})) - (b_i - x_i) = c_T(S \cup \{i\}) - (b_i - x_i). \]

Observe that in this case we have \( c_T(S \cup \{i\}) = c_T(S) \) and therefore, because \( b_i - x_i \geq 0 \) (Lemma 5.4), this yields \( c_{T^2}(S) = c_T(S \cup \{i\}) - (b_i - x_i) \leq c_T(S). \)

In case \( p \notin T(S) \), then

\[ c_{T^2}(S) = c(T(S)) - \left[ (b_i - x_i) - c(T(\{i\}) \setminus T(S)) \right]_+ \]
\[ = \min\{c(T(S)) + c(T(\{i\}) \setminus T(S)) - (b_i - x_i), c(T(S))\} \]
\[ = \min\{c(T(S \cup \{i\}) - (b_i - x_i), c(T(S))\}. \]

Hence, both cases yield, \( c_{T^2}(S) = \min\{c_T(S \cup \{i\}) - (b_i - x_i), c_T(S)\} \).

According to the claim above, we have for all \( S \subseteq N \setminus \{i\} \),

\[ w_{T^2}(S) = b(S) - c_{T^2}(S) \]
\[ = b(S) - \min\{c_T(S \cup \{i\}) - (b_i - x_i), c_T(S)\} \]
\[ = \max\{b(S \cup \{i\}) - c_T(S \cup \{i\}) - x_i, b(S) - c_T(S)\} \]
\[ = \max\{w_T(S \cup \{i\}) - x_i, w_T(S)\}. \]

As a result, we obtain for all \( S \subseteq N \setminus \{i\} \),

\[ v_{T^2}(S) := \max_{R \subseteq S} w_{T^2}(R) = \max_{R \subseteq S} \{\max_{R \subseteq S} w_T(R \cup \{i\}) - x_i, w_T(R)\} \]
\[ = \max_{R \subseteq S} \{w_T(R \cup \{i\}) - x_i, \max_{R \subseteq S} w_T(R)\} \]
\[ = \max\{v_T(S \cup \{i\}) - x_i, v_T(S)\}. \]

Finally, since \( x \in C(v_T) \) is a core allocation, we have that \( v_T(N) - x_i \geq v_T(N \setminus \{i\}) \). Hence, the equality above yields, \( v_{T^2}(N \setminus \{i\}) = v_T(N) - x_i \). This completes the proof. \[ \square \]

**Proof of Lemma 5.6.** Take \( S \in D_I(\nu) \) such that \( |S| \geq 2 \) and suppose that \( S \) is not effective. Take \( R \subseteq S \) such that \( v_T(S) = w_T(R) \). Then

\[ \text{Exc}(S, \nu) = v_T(S) - \nu(S) = w_T(R) - \nu(R) - \nu(S \setminus R) \]
\[ = v_T(R) - \nu(R) - \nu(S \setminus R) = \text{Exc}(R, \nu) - \nu(S \setminus R). \]

Note that \( \nu(S \setminus R) \geq 0 \), since \( \nu_i \geq v_T(\{i\}) = 0 \) for all \( i \in N \) (Property (P2)). Because \( \langle N, v_T \rangle \) is in particular non-negative and convex, \( \nu_i = 0 \) implies \( v_T(N) = v_T(N \setminus \{i\}) \) (Lemma 5.2). But, in a standard tree we have \( v_T(N) > v_T(N \setminus \{i\}) \) for all \( i \in N \) (Property (P4)). Therefore, we have that \( \nu(S \setminus R) > 0 \), but then the equation above yields,

\[ \text{Exc}(S, \nu) = \text{Exc}(R, \nu) - \nu(S \setminus R) < \text{Exc}(R, \nu). \]

This contradicts \( S \in D_I(\nu) \). \[ \square \]
5.4 Proofs

Proof of Lemma 5.7. Because \( \langle N, v_T \rangle \) is convex, it has a non-empty core and as a result \( E(\nu) \geq 0 \). Suppose \( E(\nu) = 0 \), then \( v_T(S) = \nu(S) \) for all \( S \in \mathcal{D}_1(\nu) \). Because \( \mathcal{D}_1(\nu) \) contains an (anti)partition (Theorem 5.3), say \( \mathcal{P} \subseteq \mathcal{D}_1(\nu) \), there exists a non-negative solution \( (\Lambda_S)_{S \in \mathcal{P}} \) of the equation \( \sum_{S \in \mathcal{P}} \Lambda_S \cdot c_S = c_N \). Hence, for this solution \( (\Lambda_S)_{S \in \mathcal{P}} \) it holds that

\[
v_T(N) = \nu(N) = \sum_{S \in \mathcal{P}} \Lambda_S \cdot v(S) = \sum_{S \in \mathcal{P}} \Lambda_S \cdot v_T(S). \tag{3}\]

Observe that if \( \{i\} \in \mathcal{P} \subseteq \mathcal{D}_1(\nu) \) then \( \nu_i = v_T(\{i\}) = 0 \) which contradicts Property (P4) (see also the proof of Lemma 5.6). Thus, according to Lemma 5.6, it follows that every \( S \in \mathcal{P} \) is effective. Combining this result with the fact that \( v_T(N) = b(N) - c_T(N) \) (Property (P4)), it is straightforward to verify that equation (3) yields,

\[
c_T(N) = \sum_{S \in \mathcal{P}} \Lambda_S \cdot c_T(S). \tag{4}\]

It is not difficult to verify (see e.g., Proposition 3.1 in Borm, Hamers and Hendrickx (2001)) that \( c_T(S) = \sum_{p \in V_0} c_p \cdot u^*_{N(B_p)}(S) \) for all \( S \subseteq N \), in which for all \( p \in V_0 \),

\[
u^*_{N(B_p)}(S) := \begin{cases} 1 & \text{if } S \cap N(B_p) \neq \emptyset, \\ 0 & \text{else,} \end{cases}
\]
i.e., \( \{u^*_{N(B_p)}\}_{p \in V_0} \) are the duals of the unanimity games \( \{u_{N(B_p)}\}_{p \in V_0} \). This yields,

\[
\sum_{S \in \mathcal{P}} \Lambda_S \cdot c_T(S) = \sum_{S \in \mathcal{P}} \Lambda_S \cdot \sum_{p \in V_0} c_p \cdot u^*_{N(B_p)}(S) = \sum_{p \in V_0} c_p \sum_{S \in \mathcal{P}} \Lambda_S \cdot u^*_{N(B_p)}(S).
\]

Because for each \( p \in V_0 \) the unanimity game \( u_{N(B_p)} \) has a non-empty core, we obtain for their duals,

\[
\sum_{S \in \mathcal{P}} \Lambda_S \cdot u^*_{N(B_p)}(S) \geq u^*_{N(B_p)}(N) \quad \text{for all } p \in V_0. \tag{5}
\]

Take \( s \in V_0 \) such that \( \pi(s) = r \). Then, due to Property (P7), we have that \( N(B_s) = N \) and therefore,

\[
\sum_{S \in \mathcal{P}} \Lambda_S \cdot u^*_{N(B_s)}(S) = \sum_{S \in \mathcal{P}} \Lambda_S > 1 = u^*_{N(B_s)}(N).
\]

Hence, one of the inequalities described in (5) is strict. Because in addition \( c_p > 0 \) for all \( p \in V_0 \) (Property (P1)) we obtain,

\[
\sum_{S \in \mathcal{P}} \Lambda_S \cdot c_T(S) = \sum_{p \in V_0} c_p \sum_{S \in \mathcal{P}} \Lambda_S \cdot u^*_{N(B_p)}(S) > \sum_{p \in V_0} c_p \cdot u^*_{N(B_p)}(N) = c_T(N).
\]

This contradicts equation (4) and completes the proof. \( \square \)

Proof of Lemma 5.8. Let \( \nu \) be the nucleolus of \( \Gamma \). Since \( v_T(\{i\}) = 0 \) (Property (P3)) and \( E(\nu) > 0 \) (Lemma 5.7) it follows that \( \nu_i > 0 \) for all \( i \in N \). Furthermore, Lemma 5.7 also implies that \( \nu(N \setminus \{i\}) > v_T(N \setminus \{i\}) \) for all \( i \in N \). Hence, for all \( i \in N \),
\[ \nu_i = v_T(N) - \nu(N \setminus \{i\}) < v_T(N) - v_T(N \setminus \{i\}) \]

\[ \leq b(N) - c_T(N) - [b(N \setminus \{i\}) - c_T(N \setminus \{i\})] = b_i. \]

The last equality uses the fact that \( c_T(N) = c_T(N \setminus \{i\}) \) for all \( i \in N \) (Property \((P_6))

Lemma 5.7 also implies that \( \nu(N(T_p)) > v_T(N(T_p)) \) for all \( p \in V_0 \) with \( \pi(p) \neq r \). As a result, we obtain,
\[ v_T(N) = \nu(N) = \nu(N(B_p)) + \nu(N(T_p)) > \nu(N(B_p)) + v_T(N(T_p)). \]

Hence,
\[ \nu(N(B_p)) < v_T(N) - v_T(N(T_p)) \]
\[ \leq b(N) - c_T(N) - [b(N(T_p)) - c(T_p)] = b(N(B_p)) - c(B_p). \]

\[ \square \]

**Proof of Proposition 5.10.** Let \( \Gamma \) be a standard tree with revenues and let \( \nu \) be the nucleolus.

(i) Take \( p \in V_0 \) with \( \pi(p) \neq r \). Note that by definition of \( E(\nu) \),
\[ \nu(N(T_p)) \geq v_T(N(T_p)) + E(\nu) \]
and \( \nu_i \geq E(\nu) \) for all \( i \in N(B_p) \).

Here we used \( v_T(\{i\}) = 0 \) for all \( i \in N \) (Property \((P_5)) \). Hence,
\[ v_T(N) = \nu(N(T_p)) + \nu(N(B_p)) \geq v_T(N(T_p)) + (|N(B_p)| + 1) \cdot E(\nu). \]

This yields,
\[ (|N(B_p)| + 1) \cdot E(\nu) \leq v_T(N) - v_T(N(T_p)) \]
\[ \leq b(N) - c_T(N) - b(N(T_p)) + c(T_p) \]
\[ = b(N(B_p)) - c(B_p). \]

Hence, \( E(\nu) \leq \alpha_p \). Furthermore, if \( \{N(T_p), \{i\}_{i \in N(B_p)}\} \subseteq D_1(\nu) \), then \( \nu(N(T_p)) = v_T(N(T_p)) + E(\nu) \) and \( \nu_i = E(\nu) \) for all \( i \in N(B_p) \). Combining this with the fact that coalition \( N(T_p) \) is effective (Lemma 5.6), yields all inequalities to be equalities. Hence, \( E(\nu) = \alpha_p \) and thus \( \nu_i = \alpha_p \) for all \( i \in N(B_p) \). Conversely, if \( E(\nu) = \alpha_p \), then the inequalities above imply that \( \{N(T_p), \{i\}_{i \in N(B_p)}\} \subseteq D_1(\nu) \).

(ii) Take a trunk \( T \subseteq V \). Again, by definition of \( E(\nu) \),
\[ v_T(N) - \nu(N(B_p)) = \nu(N(T_p)) \geq v_T(N(T_p)) + E(\nu) \]
for all \( p \in Ch(T) \) and
\[ v_T(N) - \nu_i = \nu(N \setminus \{i\}) \geq v_T(N \setminus \{i\}) + E(\nu) \]
for all \( i \in N(T) \).

Hence, by taking the sum over \( p \in Ch(T) \) and the sum over \( i \in N(T) \) we obtain,
\[ (|N(T)| + |Ch(T)|) \cdot E(\nu) \]
\[ \leq \sum_{i \in N(T)} [v_T(N) - v_T(N \setminus \{i\})] + \sum_{p \in Ch(T)} [v_T(N) - v_T(N(T_p))] - v_T(N) \]
\[ \leq b(N(T)) + \sum_{p \in Ch(T)} [b(N(B_p)) - c(B_p)] - b(N) + c_T(N) \]
\[ = c_T(N) - \sum_{p \in Ch(T)} c(B_p) = c(T). \]
Hence, $E(\nu) \leq \beta_T$. If $\{\{N(T_p)\}_{p \in CH(T)}, \{N \setminus \{i\}\}_{i \in N(T)}\} \subseteq D_1(\nu)$, then again all inequalities above are equalities and thus $E(\nu) = \beta_T$ (recall that in this case the coalitions in $\{\{N(T_p)\}_{p \in CH(T)}$ and in $\{N \setminus \{i\}\}_{i \in N(T)}$ are effective (Lemma 5.6)). Observe that in this case, we also have for all $i \in N(T)$,
$$
\nu_i = v_T(N) - v_T(N \setminus \{i\}) - \beta_T = b_i - \beta_T.
$$
Finally, if $E(\nu) = \beta_T$, the above inequalities imply that
$$
\{\{N(T_p)\}_{p \in CH(T)}, \{N \setminus \{i\}\}_{i \in N(T)}\} \subseteq D_1(\nu).
$$

(iii) Take $i \in N$. Then $\nu(N \setminus \{i\}) \geq v_T(N \setminus \{i\}) + E(\nu)$ and $\nu_i \geq E(\nu)$. Hence,
$$
2\cdot E(\nu) \leq v_T(N) - v_T(N \setminus \{i\}) \leq b_i + c_T(N \setminus \{i\}) - c_T(N) = b_i.
$$
The last equality follows from Property (P_6). As a result, $E(\nu) \leq \gamma_i$. Furthermore, if $\{\{N \setminus \{i\}\}, \{i\}\} \subseteq D_1(\nu)$, then coalition $\{N \setminus \{i\}\}$ is effective and thus $E(\nu) = \gamma_i$ which on its turn implies that $\nu_i = \gamma_i$. Conversely, if $E(\nu) = \gamma_i$, then the inequalities above imply that $\{\{N \setminus \{i\}\}, \{i\}\} \subseteq D_1(\nu)$.

(iv) Observe that $E(\nu) \leq \nu_i$ for all $i \in N$ and therefore, $E(\nu) \leq \nu(N) = v_T(N)$. Hence, $E(\nu) \leq \delta$. From this it immediately follows that $\{\{i\}\}_{i \in N} \subseteq D_1(\nu)$ if and only if $E(\nu) = \delta$. Moreover, $E(\nu) = \delta$ implies $\nu_i = \delta$ for all $i \in N$. \qed
5 The Nucleolus of Trees with Revenues
6 Processing Games

6.1 Introduction

This chapter analyzes processing problems and related cooperative games. It focuses on the non-emptiness of the core of these cooperative TU-games. The chapter is based on Meertens, Borm, Reijnierse and Quant (2004).

Consider the situation in which a number of jobs have to be completed, each requiring its own amount of effort and in which there is a capacity constraint to process jobs. The terminology has been chosen very general in order to allow several interpretations. For instance, jobs can involve maintenance tasks, the manufacturing of products, computational tasks or investments under periodic budget raises. Capacity constraints can be induced by limited availability of labor and/or engine power, by periodic supplies of raw material, by maximum computational speed of a computer facility or by budget. In these examples, effort represents performance of men and/or machinery, volumes of raw material, calculations or money. In all cases, capacity means the maximum effort per time unit. It is assumed that for each time unit that a job is uncompleted, a fixed cost (called the cost-coefficient of the job) has to be paid. There are no restrictions on the processing schedule with respect to, for instance, pre-emption, semi-active, or serial vs. parallel planning. Given the capacity constraint, the objective is to find a schedule for completing all jobs in such a way that the total costs are minimal. We have baptized this problem as a processing problem.

In the first part of the chapter, we prove that in order to minimize the total costs in a processing problem, the jobs have to be completed one by one. So, until all jobs have been completed, all capacity should be used for one job at a time. As a result, it suffices to find an optimal order of the jobs. This can be done by applying Smith’s rule (Smith (1956)), i.e., the jobs are handled in the order of decreasing urgencies. The urgency of a job is defined to be its cost-
Processing Games

coefficient divided by its processing demand. From this observation it follows that processing problems and sequencing problems with one machine and aggregated weighted completion times (see e.g., French (1982)) are equivalent. The main characteristics of such a sequencing problem is that a number of jobs have to be processed in some order on exactly one machine in such a way that the aggregated weighted completion-times are minimal.

However, the problems diverge when analyzed in a cooperative game theory framework. Problems are extended to situations in which each job belongs to a (different) player and each player has a personal capacity to handle jobs. Besides minimizing total costs, costs have to be allocated to each player individually. In order to find fair allocations a cooperative game is constructed. Sequencing problems with one machine have also been analyzed from a game-theoretical point of view in several ways, starting from the basic paper by Curiel, Pederzoli and Tijs (1989) (see Curiel, Hamers and Klijn (2002) for an overview). In the present chapter we associate cooperative games to processing situations. These games are called processing games and differ from sequencing games. This diversion is due to two main differences between a processing and a sequencing situation. The first difference is that in a processing situation the players have individual, and generally speaking, different capacities to handle jobs (not necessarily their own), while in a sequencing situation with one machine there are no individual capacities. In fact, the machine processes all jobs with a constant capacity. The second difference is that in a processing situation there is no fixed initial order in which the jobs stand in line in front of a machine. So, in a processing situation there are no initial restrictions nor rights on the order in which players may process their jobs. Here, we should also mention the work of Maniquet (2003), who studies a model in between. He considers queueing problems in which there is also no fixed initial order on the jobs. However, unlike our model, the capacity to process jobs is fixed.

Let us elaborate on the way players can cooperate in processing situations. If a coalition is formed, costs savings can be made by helping each other by means of using a player’s capacity to speed up the completion of a job of another coalition member. To put it differently, the members of the coalition have to their disposal the sum of their individual capacities for completing all jobs of the coalition. This situation can be modeled as a processing problem and as a result one can determine an optimal schedule and its costs. However, the problem of minimizing the total costs is supplemented with the problem of dividing these costs among the players involved. The latter is of a typical game theoretical nature and in order to solve it, we analyze the complete processing game with respect to core elements. Here, a processing game is a cooperative cost-game, in which the costs of a coalition equals the costs of an optimal schedule of its corresponding processing problem.

The main result of this chapter, presented in Section 6.3, states that every processing game is totally balanced, i.e., every subgame of a processing game has a non-empty core. Intuitively, this means that if a group of players decides to cooperate, their total minimal costs can be divided in a fair way with respect to a core allocation. To prove this statement, we construct from a given processing situation an exchange economy with land. In this Debreu-type (Debreu (1959)) of exchange economy each player initially owns a part of a perfectly divisible two dimensional commodity, referred to as land. One dimension is time and the other one is effort per time unit. In the context of processing situations, one can interpreted this com-
modity as an agenda. In order to complete their jobs, players must make reservations in the agenda, i.e., a player must book a block of time and effort per time unit which is sufficiently large to complete his job. An interpretation is that during this reserved period of time, he has the booked effort per time unit to his disposal for completing his job. A price is introduced such that the market clears, i.e., no part of the agenda is booked by more than one player. Clearing the market will, as usual, lead to a price equilibrium (Walras Law). From this price equilibrium we construct an allocation contained in the core of the processing game.

So, we explicitly provide a core allocation for every processing game. Since a subgame of a processing game is another processing game, we obtain total balancedness. Furthermore, an interpretation of this core allocation is included along with a proof that it is independent of which optimal schedule is chosen by a coalition to process their jobs. Differently, if for a coalition of players there are several optimal schedules, the contribution a player has in the total costs is equal in any of the optimal orders.

Finally, in Section 6.5, we briefly discuss a generalization of processing situations, as can be found in Quant, Meertens and Reijnierse (2004).

### 6.2 Processing problems

A processing problem $\mathcal{P}$ can be described in a formal way by the tuple

$$\langle J, (p_j)_{j \in J}, (\alpha_j)_{j \in J}, \beta \rangle.$$

Here, $J$ is a finite set of jobs that need to be completed. Each job $j \in J$ has a processing demand $p_j \in \mathbb{R}_+$. Furthermore, $\alpha \in \mathbb{R}_+^J$ is the vector of cost-coefficients and $\beta > 0$ is a strictly positive real number denoting the maximum effort per time unit, or shortly capacity. During a period of time $t$ that a job $j \in J$ is uncompleted its costs equal $\alpha_j \cdot t$. A feasible schedule to process the jobs in $J$ can be described by a map

$$F: J \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

with the following properties:

(i) $F(j, \cdot): \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous and weakly increasing for all $j \in J$,

(ii) $\sum_{j \in J} \left[ F(j, t) - F(j, s) \right] \leq \beta \cdot (t - s)$ for every $s, t \in \mathbb{R}_+$ with $s \leq t$.

The value $F(j, t)$ for job $j \in J$ at time $t \in \mathbb{R}_+$ can be interpreted as the cumulative amount of effort which has been used for job $j \in J$ up to time $t$. Observe that a job $j \in J$ is completed as soon as the value $F(j, t)$ exceeds the processing demand $p_j$. Property (ii) states that for each segment $[s, t]$ of time, the total effort spend on all jobs together is restricted linearly in the length of the segment by the capacity constraint. We denote $\mathcal{F}$ as the family of all feasible schedules.

Given a feasible schedule $F \in \mathcal{F}$, we define $T_j(F)$ as the completion-time of job $j \in J$, i.e.,

$$T_j(F) := \inf \{ t \in \mathbb{R}_+ \mid F(j, t) \geq p_j \}.$$
We allow $T_j(F)$ to be infinity. The objective is to complete all jobs such that the sum of their costs is minimized. This minimum is expressed by

$$c(\mathcal{P}) := \inf_{F \in \mathcal{F}} \sum_{j \in J} \alpha_j T_j(F).$$

**Example.** Suppose a farmer has to harvest three acres with different types of crop, say types 1, 2 and 3. The tasks require 20, 30 and 10 days of work for one man, respectively. His available workforce consists of himself and five more employees. He has contracts with distributors to deliver the types of crop, but he is already over time. Every extra day of delay results in penalties of size 24, 30 and 6, respectively. The farmer wants to harvest the acres in such a way that the total sum of penalties will be minimal. This problem can be modeled as the processing problem $\mathcal{P} := (J, p, \alpha, \beta)$, in which $J := \{1, 2, 3\}$, $p := (20, 30, 10)$, $\alpha := (24, 30, 6)$ and $\beta := 6$.

One approach to complete the jobs is by dividing the capacity $\beta$ over the jobs, for instance, proportionally to their processing demands. Then after 10 days all jobs are finished simultaneously.

This approach corresponds with the schedule $F \in \mathcal{F}$ defined by

$$F(1, t) := 2 \cdot t, \quad F(2, t) := 3 \cdot t \quad \text{and} \quad F(3, t) := t \quad \text{for all} \quad t \in \mathbb{R}_+.$$

It yields a total cost of $10 \cdot (24 + 30 + 6) = 600$. Another approach is to use the total capacity on exactly one job and finish the jobs one after another, for instance, first job 1, then job 2 and finally job 3. This approach is depicted in the following figure.
6.2 Processing problems

In this second approach, the corresponding schedule \( F' \in \mathcal{F} \) equals

\[
\begin{align*}
F'(1,t) & := \begin{cases} 
6 \cdot t & \text{if } 6 \cdot t \in [0, 20], \\
20 & \text{if } 6 \cdot t \geq 20,
\end{cases} \\
F'(2,t) & := \begin{cases} 
0 & \text{if } 6 \cdot t \in [0, 20], \\
6 \cdot t - 20 & \text{if } 6 \cdot t \in [20, 50], \\
30 & \text{if } 6 \cdot t \geq 50,
\end{cases} \\
F'(3,t) & := \begin{cases} 
0 & \text{if } 6 \cdot t \in [0, 50], \\
6 \cdot t - 50 & \text{if } 6 \cdot t \in [50, 60], \\
10 & \text{if } 6 \cdot t \geq 60.
\end{cases}
\end{align*}
\]

The schedule \( F' \) induces completion times \( T(F') = (\frac{20}{6}, \frac{50}{6}, 10) \) which yield a total cost of \( \frac{20}{6} \cdot 24 + \frac{50}{6} \cdot 30 + 10 \cdot 6 = 390 \). Hence, the latter approach is more profitable for the farmer than the previous one.

In the previous example it is already stressed out that it may be profitable to use the total capacity for exactly one job and finish the jobs one after another. We start by demonstrating that in order to minimize the total costs in an arbitrary processing problem, one should indeed choose this approach. To do so, we first need some notational conventions.

Let \( \sigma: \{1, \ldots, |J|\} \rightarrow J \) be a bijection. It can be seen as the order in which the jobs in \( J \) are completed, i.e., the job at position \( i \) in the order \( \sigma \) is denoted by \( \sigma(i) \). We write \( \Pi(J) \) for the family of all bijections from \( \{1, \ldots, |J|\} \) to \( J \). If the jobs in \( J \) are completed in the order 2, then we get as corresponding schedule \( F^\sigma: J \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) defined by

\[
F^\sigma(\sigma(j), t) := \begin{cases} 
0 & \text{if } \beta \cdot t \leq \sum_{k<j} p_{\sigma(k)}, \\
\beta \cdot t - \sum_{k<j} p_{\sigma(k)} & \text{if } \sum_{k<j} p_{\sigma(k)} \leq \beta \cdot t \leq \sum_{k<j} p_{\sigma(k)}, \\
\beta \cdot t & \text{if } \sum_{k<j} p_{\sigma(k)} \leq \beta \cdot t.
\end{cases}
\]

It is straightforward to verify that \( F^\sigma \) is a feasible schedule. Furthermore, given a feasible schedule \( F^\sigma \in \mathcal{F} \), the completion time for every job is finite. The following lemma states that there exists an order \( \sigma \in \Pi(J) \) such that \( F^\sigma \) is an optimal schedule.

**Lemma 6.1.** There exists a bijection \( \sigma \in \Pi(J) \) such that \( \sum_{j \in J} \alpha_j T_j(F^\sigma) = c(\mathcal{P}) \).

**Proof.** Let \( F \in \mathcal{F} \) be a feasible schedule such that \( T_j(F) < \infty \) for all \( j \in J \). Let \( \sigma \in \Pi(J) \) be a bijection such that

\[
T_{\sigma(1)}(F) \leq T_{\sigma(2)}(F) \leq \ldots \leq T_{\sigma(|J|)}(F).
\]

Take \( 1 \leq i \leq |J| \), then we have,

\[
T_{\sigma(i)}(F) \geq \frac{i}{r} \sum_{k=1}^i p_{\sigma(k)} = T_{\sigma(i)}(F^\sigma).
\]
This yields,

\[
\sum_{i=1}^{\vert J \vert} \alpha_{\sigma(i)} \cdot T_{\sigma(i)}(F) \geq \sum_{i=1}^{\vert J \vert} \alpha_{\sigma(i)} \cdot T_{\sigma(i)}(F^\sigma).
\]

(1)

So, for every feasible schedule \( F \in \mathcal{F} \), there exists a feasible schedule \( F^\sigma \in \mathcal{F} \) such that inequality (1) holds. Hence, \( c(\mathcal{P}) = \min_{\sigma \in \Pi(J)} \sum_{i \in J} \alpha_i \cdot T_i(F^\sigma) \).

Lemma 6.1 shows that if the above cost criterion is considered, a processing problem boils down to a sequencing problem with one machine and aggregated weighted completion times (see e.g., French (1982)). In this problem, a finite number of jobs with certain processing times need to be completed on one machine such that the aggregated weighted completion-times are minimal. By defining the processing time of a job \( j \in J \) as its processing demand \( p_j \) divided by the capacity constraint \( \beta \), one can transform a processing problem into such a sequencing problem. Therefore, the optimal schedule for a processing problem can be found by applying Smith’s rule (Smith (1956)), i.e., process the jobs in the order of decreasing urgencies, in which the urgency of job \( j \in J \) is given by \( \frac{p_j}{\beta} \). For the sake of completeness we give a proof of this statement.

Proposition 6.2 (Smith (1956)). Let \( \mathcal{P} \) be a processing problem in which the jobs \( J = \{1, \ldots, m\} \) are numbered such that \( \frac{p_1}{p_1} \geq \cdots \geq \frac{p_m}{p_m} \). Then it is optimal to process the jobs in increasing order and

\[
c(\mathcal{P}) = \sum_{k=1}^{m} \alpha_k \cdot (p_1 + \ldots + p_k).
\]

Proof. According to Lemma 6.1 there exists a bijection \( \sigma \in \Pi(J) \) such that

\[
c(\mathcal{P}) = \sum_{j \in J} \alpha_j \cdot T_j(F^\sigma) = \frac{1}{\beta} \sum_{k=1}^{m} \alpha_{\sigma(k)} \cdot \sum_{\ell \leq k} p_{\sigma(\ell)}.
\]

We have to show that \( \sigma \) can be chosen to be the identity. Let us first assume that even \( \frac{p_1}{p_1} > \cdots > \frac{p_m}{p_m} \). Suppose that \( \frac{p_{\sigma(k+1)}}{p_{\sigma(k)}} \geq \frac{p_{\sigma(k)}}{p_{\sigma(k)}} \) for a certain number \( k \) with \( 1 \leq k < m \). Let \( \tau \in \Pi(J) \) be the bijection such that \( \tau(k) = \sigma(k+1), \tau(k+1) = \sigma(k) \) and \( \tau(\ell) = \sigma(\ell) \) for all \( \ell \neq k, k+1 \).

By the optimality of \( \sigma \) we have that

\[
0 \geq \sum_{j \in J} \alpha_j \cdot T_j(F^\sigma) - \sum_{j \in J} \alpha_j \cdot T_j(F^\tau)
\]

\[
= \frac{\alpha_{\sigma(k)}}{\beta} \left( \sum_{\ell=1}^{k-1} p_{\sigma(\ell)} + p_{\sigma(k)} \right) + \frac{\alpha_{\sigma(k+1)}}{\beta} \left( \sum_{\ell=1}^{k-1} p_{\sigma(\ell)} + p_{\sigma(k)} + p_{\sigma(k+1)} \right)
\]

\[
= \frac{\alpha_{\tau(k)}}{\beta} \left( \sum_{\ell=1}^{k-1} p_{\tau(\ell)} + p_{\tau(k)} \right) - \frac{\alpha_{\tau(k+1)}}{\beta} \left( \sum_{\ell=1}^{k-1} p_{\tau(\ell)} + p_{\tau(k)} + p_{\tau(k+1)} \right).
\]
6.3 Processing games

\[ \begin{align*}
&= \frac{\alpha_{\sigma(k)}}{\beta} \left( \sum_{\ell=1}^{k-1} p_{\sigma(\ell)} + p_{\sigma(k)} \right) + \frac{\alpha_{\sigma(k+1)}}{\beta} \left( \sum_{\ell=1}^{k-1} p_{\sigma(\ell)} + p_{\sigma(k)} + p_{\sigma(k+1)} \right) \\
&\quad - \frac{\alpha_{\sigma(k+1)}}{\beta} \left( \sum_{\ell=1}^{k-1} p_{\sigma(\ell)} + p_{\sigma(k+1)} \right) - \frac{\alpha_{\sigma(k)}}{\beta} \left( \sum_{\ell=1}^{k-1} p_{\sigma(\ell)} + p_{\sigma(k)} + p_{\sigma(k+1)} \right) \\
&= \frac{1}{\beta} \left( \alpha_{\sigma(k+1)} p_{\sigma(k)} - \alpha_{\sigma(k)} p_{\sigma(k+1)} \right) > 0.
\]

This contradiction shows that \( \frac{\alpha_{\sigma(k+1)}}{p_{\sigma(k+1)}} \leq \frac{\alpha_{\sigma(k)}}{p_{\sigma(k)}} \) for all \( 1 \leq k \leq m \). By the assumption that \( \frac{\alpha_1}{p_1} > \cdots > \frac{\alpha_m}{p_m} \), the bijection \( \sigma \in \Pi(J) \) must be the identity. A continuity argument shows that \( \sigma \) still can be chosen to be the identity if we weaken this assumption to \( \frac{\alpha_1}{p_1} \geq \cdots \geq \frac{\alpha_m}{p_m} \).

So, Proposition 6.2 provides a method to find for every processing problem an optimal schedule. In the next section we look at processing problems from a game-theoretical point of view. Processing problems are extended to situations in which a player, endowed with a personal capacity, is obliged to complete a job. In this case, the problem of minimizing the total costs is supplemented with the problem of dividing these costs among the players in a fair way.

6.3 Processing games

In a processing situation \( \langle N, J, (p_j)_{j \in J}, (\alpha_j)_{j \in J}, (\beta_j)_{j \in N} \rangle \) there is a finite set of players \( N \), in which each player \( i \in N \) is equipped with a strictly positive (individual) capacity \( \beta_i \) to perform jobs in \( J \). Again, each job \( j \in J \) has processing demand \( p_j \in \mathbb{R}_+ \) and cost-coefficient \( \alpha_j \in \mathbb{R}_+ \). As long as a job \( j \) is uncompleted, it generates a cost of size \( \alpha_j \) per time unit. Each player has to complete one specific job in \( J \). Since, each player is obliged to a different job, there is a one-one correspondence between players and jobs. Therefore, for simplicity, we denote the processing demand and cost-coefficient of the job of player \( i \), by \( p_i \) and \( \alpha_i \), respectively.

Let \( S \subseteq N \) be a coalition which decides to cooperate. This coalition has the disposal of the individual capacities of all of its members to perform jobs, i.e., coalition \( S \) can maximally generate an amount of effort of size \( \beta(S) := \sum_{j \in S} \beta_j \) per time unit. The aim of coalition \( S \) is to complete all jobs of its members such that the aggregate costs are minimized. This gives rise to a processing problem

\[ P(S) := \langle J(S), (p_i)_{i \in S}, (\alpha_i)_{i \in S}, \beta(S) \rangle, \]

in which \( J(S) \) denotes the set of jobs to be completed by the players in \( S \). Proposition 6.2 provides a method to calculate an optimal schedule such that the aggregate costs of coalition \( S \) are minimized. However, constructing such a schedule is only a part of the problem. That is, in addition to minimizing total costs, the problem remains how to allocate in a fair way these costs among the players in \( S \). To analyze this problem, we construct the processing game \( \langle N, c_P \rangle \) with \( c_P : 2^N \rightarrow \mathbb{R}_+ \) defined by

\[ c_P(S) := c(P(S)) \text{ for all } S \subseteq N. \]
The processing game \( \langle N, c_P \rangle \) is a so-called cost-game. A cost-game is a TU-game like we studied in Chapter 2, but instead of profits it considers costs to coalitions. So, the main goal of a coalition, when it forms, is to minimize its total costs. Because of the different interpretation of the definitions of solution concepts for TU-games as well as properties on TU-games have to be adjusted to this context (see Section 2.2). For instance, the core of a cost-game \( \langle N, c \rangle \) is defined by

\[
\mathcal{C}(c) := \{ x \in \mathbb{R}^N \mid x(N) = c(N) \text{ and } x(S) \leq c(S) \text{ for all } S \subseteq N \}.
\]

In a similar way the definition of the imputation set \( I(c) \) is altered. A cost-game \( \langle N, c \rangle \) is said to be balanced, or equivalently has a non-empty core (Theorem 2.1) if for every non-negative solution \( \langle \lambda_S \rangle_{S \subseteq N} \) of 

\[
\sum_{S \subseteq N} \lambda_S \cdot c(S) = c(N)
\]

the inequality 

\[
\sum_{S \subseteq N} \lambda_S \cdot c(S) \geq c(N)
\]

holds. Recall from Section 2.2, that a TU-game is said to be totally balanced whenever every subgame has a non-empty core. Finally, we mention the alternative definitions of superadditivity and convexity for cost-games. The counterpart of superadditivity is subadditivity, i.e., a cost-game \( \langle N, c \rangle \) is said to be subadditive if 

\[
c(S) + c(T) \geq c(S \cup T)
\]

whenever every subgame has a non-empty core. And the counterpart of convexity is concavity, i.e., a cost-game \( \langle N, c \rangle \) is concave if 

\[
c(S) + c(T) \geq c(S \cup T) + c(S \cap T)
\]

for all \( S, T \subseteq N \). Similarly, like in case of TU-games with rewards, concavity of a cost-game implies (total) balancedness.

The main goal of this chapter is to prove that a processing game is totally balanced. This means, we consider the core as a normative concept for an allocation of the total costs to be fair, i.e., no coalition can improve with respect to this allocation. Before we do so, subadditivity is shown firstly.

**Proposition 6.3.** Processing games are subadditive.

**Proof.** Let \( \langle N, J, p, \alpha, \beta \rangle \) be a processing situation and let \( S \) and \( T \) be two disjoint coalitions. Let \( F_S \) and \( F_T \) be optimal schedules for the processing problems \( \mathcal{P}(S) \) and \( \mathcal{P}(T) \), respectively. It is straightforward to check that 

\[
F_{S \cup T}(j, t) := \begin{cases} 
F_S(j, t) & \text{if } j \in J(S), \\
F_T(j, t) & \text{if } j \in J(T),
\end{cases}
\]

is a feasible schedule for coalition \( S \cup T \). Therefore,

\[
c_P(S \cup T) \leq \sum_{j \in J(S \cup T)} \alpha_j \cdot T_j(F_{S \cup T})
\]

\[
= \sum_{j \in J(S)} \alpha_j \cdot T_j(F_S) + \sum_{j \in J(T)} \alpha_j \cdot T_j(F_T)
\]

\[
= c_P(S) + c_P(T).
\]

Hence, the processing game \( \langle N, c_P \rangle \) is subadditive. \( \square \)

Next, we revisit the example of the previous subsection to show a processing game. The example points out that a processing game is in general not concave and that there can be players that are attained a negative cost (i.e., a reward) in any core allocation. As a result, solutions based on a proportional type of costs allocation with respect to, for instance, processing times and/or capacities will not generate core allocations in general.
6.3 Processing games

Example. This time the three acres of the previous example are owned by three different farmers. Farmers 1 and 2 have small farms and no employees. Farmer 3 has three employees. If the farmers decide to join forces, a processing game can help to share the costs savings fairly.

In the processing situation comporting with the story, the player set \( N \) consists of the players 1, 2 and 3 of which the respective jobs \( j_1, j_2, j_3 \) form \( J \). The processing demands of the jobs are given by \( p := (20, 30, 10) \), the cost-coefficients are given by \( \alpha := (24, 30, 6) \) and the individual capacities of the players are \( \beta := (1, 1, 4) \). Observe that the players are numbered in such a way that \( \frac{24}{p_1} \geq \frac{24}{p_2} \geq \frac{24}{p_3} \). According to Proposition 6.2, the corresponding processing game \( (N, c_P) \) is given by:

\[
\begin{align*}
  c_P(N) &= \frac{1}{3} \cdot (24 \cdot 20 + 30 \cdot (20 + 30) + 6 \cdot (20 + 30 + 10)) = 390, \\
  c_P(\{1,2\}) &= \frac{1}{3} \cdot (24 \cdot 20 + 30 \cdot 50) = 990, \\
  c_P(\{1\}) &= 480, \\
  c_P(\{2\}) &= 900 \text{ and } c_P(\{3\}) = 15.
\end{align*}
\]

Observe that \( c_P(N) + c_P(\{2\}) = 1290 > 1218 = c_P(\{1,2\}) + c_P(\{2,3\}) \). So, the cost-game \( (N, c_P) \) is not concave. Furthermore, if \( x \in C(c_P) \), then \( x_3 < 0 \). Indeed, if \( x \in C(c_P) \), then in particular we have the following inequalities:

\[
\begin{align*}
  x_1 + x_3 &\leq 132, \\
  x_2 + x_3 &\leq 228, \\
  x_1 + x_2 + x_3 &= 390.
\end{align*}
\]

Hence, \( 390 + x_4 = x_1 + x_2 + 2 \cdot x_3 \leq 360 \). As a result, this yields \( x_3 \leq -30 \). Note that player 3 is rewarded for his participation in every core allocation because of his relatively large capacity.

It is left to the reader to verify that the allocation \( (195, 310, -115) \) is contained in the core \( C(c_P) \).

The core allocation of the example above has been found by applying the following theorem which is the main result of this chapter.

**Theorem 6.4.** Processing games are totally balanced. \( \square \)

A proof for Theorem 6.4 can be found in Section 6.4. In fact, there it is demonstrated that the vector \( X \in \mathbb{R}^N \) defined for all \( i \in N \) by

\[
X_i := \frac{\alpha_i}{\beta_i^{(N)}} \sum_{k=1}^{i} p_k + \frac{\beta_i}{\beta_i^{(N)}} (\sum_{k=1}^{n} \alpha_k) - \frac{\beta_i}{\beta_i^{(N)}} \sum_{k=1}^{n} \frac{p_k}{\beta_i^{(N)}} (\sum_{\ell=k+1}^{n} \alpha_k + \sum_{\ell}^{n} \alpha_{\ell})
\]

is a core allocation of the processing game \( (N, c_P) \), provided that \( N := \{1, \ldots, n\} \) and \( \frac{p_i}{\beta_i} \geq \ldots \geq \frac{p_n}{\beta_n} \).

Let us give an interpretation of the described core allocation. Because of the assumption that urgencies are ordered in the way described above, it is optimal for the grand coalition \( N \) to...
work all together on the job of player 1, then on the job of player 2 and so on (Proposition 6.2). According to this schedule player \( i \) has to wait for a period of time with length \( \frac{1}{\beta(N)} \sum_{k=1}^{\ell} p_{k} \) until his job has been completed. As a result, his individual direct costs will be

\[ D_i := \alpha_i \cdot \frac{1}{\beta(N)} \sum_{k=1}^{i} p_k. \]  

(2)

If each player \( i \in N \) would pay this amount, the costs are divided in an efficient way. It would not be very fair though. A player whose job is placed at the end of the line should be compensated. Furthermore, players who have a relatively large capacity \( \beta_i \) should be rewarded. This can be done as follows. Besides the direct costs (2), a tax is introduced on the jobs. The tax proceeds will then be used to subsidize the players with large capacities. More particularly, the sum of the tax deposits is redivided proportionally to the capacities of the players. Let us explain the reasoning behind the explicit format of the tax deposits. At each moment of time \( t \), a cost-rate is introduced. The player whose job is in process must pay this rate. The cost-rate at time \( t \) equals

\[ \frac{\beta_k}{\beta(N)} \cdot (\text{the proportion of job } j_k \text{ that has not been finished yet at time } t). \]

During a period of time with length \( \frac{p_{i_k}}{\beta(N)} \) all players are working on the job of player \( i \). Player \( i \) must pay \( \alpha_i \cdot \frac{p_{i_k}}{\beta(N)} \) for each player \( k \) whose job is still waiting to be processed. This is exactly the loss of player \( i \), because of the fact that the job of player \( i \) is processed before his own job. Additionally, player \( i \) has to pay \( \frac{1}{2} \cdot \alpha_i \cdot \frac{p_{i_k}}{\beta(N)} \), since the mean proportion of his own job that has not been finished yet during its processing time equals \( \frac{1}{2} \). The sum of these amounts make the tax deposit of player \( i \) to be

\[ T_i := \frac{\beta_i}{\beta(N)} \cdot \left( \frac{1}{2} \cdot \alpha_i + \sum_{k=1}^{n} \alpha_k \right). \]

(3)

Finally, the total amount of collected tax money is returned to the players, proportional to their individual capacities. This yields a subsidy for player \( i \) of

\[ S_i := \frac{\alpha_i}{\beta(N)} \sum_{k=1}^{n} T_k = \frac{\beta_i}{\beta(N)} \sum_{k=1}^{n} \frac{p_k}{\beta(N)} \cdot \left( \frac{1}{2} \cdot \alpha_k + \sum_{\ell=k+1}^{n} \alpha_{\ell} \right). \]

(4)

Subtracting expression (4) from the sum of the expressions (2) and (3), yields the amount player \( i \in N \) has to pay according to the core allocation \( X \), i.e.,

\[ X_i := D_i + T_i - \frac{\beta_i}{\beta(N)} \sum_{k=1}^{n} T_k \text{ for all } i \in N. \]

(5)

Let us return once more to the processing situation arising from the example with the three farmers.

**Example.** Let \((N, J, p, \alpha, \beta)\) be the processing situation with \( N := \{1, 2, 3\} \), \( J := \{j_1, j_2, j_3\} \), \( p := (20, 30, 10) \), \( \alpha := (24, 30, 6) \) and \( \beta := (1, 1, 4) \).
6.3 Processing games

We already stressed out that the allocation \((195, 310, -115)\) is a core allocation of the corresponding processing game. This allocation arises as follows. It consists of the individual direct costs (expression (2))

\[
D_1 = 24 \cdot \frac{1}{6} \cdot 20 = 80, \quad D_2 = 30 \cdot \frac{1}{6} \cdot 50 = 250 \quad \text{and} \quad D_3 = 6 \cdot \frac{1}{6} \cdot 60 = 60.
\]

Observe that the allocation \((D_1, D_2, D_3)\) is indeed efficient, but clearly not a core element of the processing game \(\langle N, c_p \rangle\).

The tax that the players have to pay is (expression (3))

\[
T_i = \frac{w_i}{h} \cdot (\frac{1}{x} \cdot 24 + 30 + 6) = 160, \quad T_2 = \frac{w_i}{h} \cdot (\frac{1}{x} \cdot 30 + 6) = 105 \quad \text{and} \quad T_3 = \frac{w_i}{h} \cdot (\frac{1}{x} \cdot 6) = 5.
\]

According to expression (4) the players are subsidized with

\[
S_1 = \frac{1}{6} \cdot 270 = 45, \quad S_2 = \frac{1}{6} \cdot 270 = 45 \quad \text{and} \quad S_3 = \frac{1}{6} \cdot 270 = 180,
\]

respectively. Hence, according to expression (5), the allocation \((D_1 + T_1 - S_1, D_2 + T_2 - S_2, D_3 + T_3 - S_3) = (195, 310, -115)\) is contained in the core \(C(c_p)\).

Observe that the direct costs as well as the tax deposits are based on the given optimal order of decreasing urgencies. At first sight, the core allocation given in (5) depends therefore on the optimal order chosen. The following proposition shows that this is an optical illusion.

**Proposition 6.5.** Let \(\langle N, J, p, \alpha, \beta \rangle\) be a processing situation. The core allocation \(X\), given in (5), does not depend on the choice of which optimal order is used to process the jobs.

**Proof.** Two optimal orders can be obtained from each other by a series of switches of two adjacent jobs with equal urgencies (Proposition 6.2). It is sufficient to show that \(X\) does not change at each of these switches. Assume that one optimal order is \((1, \ldots, n)\) and that players \(i\) and \(i + 1\) have equal urgencies, i.e., \(\frac{\alpha_i}{p_i} = \frac{\alpha_{i+1}}{p_{i+1}}\). We have to show that \(X\) and \(X'\) coincide, with \(X\) and \(X'\) denoting the costs allocations which correspond to the order \((1, \ldots, n)\) and to the order \((1, \ldots, i - 1, i + 1, i, i + 2, \ldots, n)\), where \(i\) and \(i + 1\) have been switched, respectively. The vectors of tax deposits corresponding to these orders are denoted by \(T\) and \(T'\), respectively.

We first show that the total amount of taxes paid in both orders is the same. Note that for players \(k\) unequal to \(i\) and \(i + 1\) the taxes \(T_k\) and \(T'_k\) coincide. It is shown below that the sum of the taxes paid by \(i\) and \(i + 1\) does not change either.

\[
T_i + T_{i+1} = \frac{w_i}{h} \cdot \left( \frac{1}{x} \cdot \alpha_i + \sum_{\ell=i+2}^{n} \alpha_\ell \right) + \frac{w_{i+1}}{h} \cdot \left( \frac{1}{x} \cdot \alpha_{i+1} + \sum_{\ell=i+2}^{n} \alpha_\ell \right)
\]

\[
= \frac{w_i}{h} \cdot \left( \frac{1}{x} \cdot \alpha_i + \sum_{\ell=i+2}^{n} \alpha_\ell \right) + \frac{w_{i+1}}{h} \cdot \left( \frac{1}{x} \cdot \alpha_{i+1} + \sum_{\ell=i+2}^{n} \alpha_\ell \right)
\]

\[
= \frac{w_i}{h} \cdot \left( \frac{1}{x} \cdot \alpha_i + \sum_{\ell=i+2}^{n} \alpha_\ell \right) + \frac{w_{i+1}}{h} \cdot \left( \frac{1}{x} \cdot \alpha_{i+1} + \sum_{\ell=i+2}^{n} \alpha_\ell \right)
\]

\[
= T_i' + T_{i+1}'.
\]
The third equality follows from the fact that the job of player $i$ and the job of player $i + 1$ have equal urgencies. So, the total sum of amount of taxes is equal in both orders. Since, the taxes for players unequal to $i$ and $i + 1$ in both orders coincide, it is immediately clear that $X_k = X'_k$ for all $k \neq i, i + 1$. We now prove that $X_i = X'_i$.

$$X_i = \frac{\alpha_i}{\beta N} \sum_{k=1}^{i} p_k + T_i - \frac{\beta_i}{\beta N} \sum_{k=1}^{n} T_k$$

$$= \frac{\alpha_i}{\beta N} \sum_{k=1}^{i} p_k + \frac{p_i}{\beta N} \alpha_{i+1} + \frac{p_i}{\beta N} \left( \frac{1}{2} \alpha_i + \sum_{k=i+2}^{n} \alpha_k \right) - \frac{\beta_i}{\beta N} \sum_{k=1}^{n} T'_k$$

$$= \frac{\alpha_i}{\beta N} \sum_{k=1}^{i} p_k + \frac{p_i}{\beta N} \alpha_i + T'_i - \frac{\beta_i}{\beta N} \sum_{k=1}^{n} T'_k$$

$$= \frac{\alpha_i}{\beta N} \left( \sum_{k=1}^{i} p_k + p_{i+1} \right) + T'_i - \frac{\beta_i}{\beta N} \sum_{k=1}^{n} T'_k = X'_i.$$

The third equality uses the fact that the job of player $i$ and the job of player $i + 1$ have the same urgency. In the same way it can be proved that $X_{i+1}$ and $X'_{i+1}$ coincide. \(\square\)

So, in case a processing situation admits more than one optimal order, the contribution to the total costs by the players, according to the core allocation given in (5), is independent of which optimal schedule is chosen. Because of this result, the allocation $X$, given in (5), can be considered to be continuous in $\alpha$, $\beta$ and $p$. It is clear that it is continuous at points with just one optimal order. Proposition 6.5 shows that it is also continuous in points $(\alpha, \beta, p)$ which generate more than one optimal order. As a result, we obtain the following corollary.

**Corollary 6.6.** The core allocation $X$, given in (5), is continuous in $\alpha$, $\beta$ and $p$. \(\square\)

### 6.4 Proof of Theorem 6.4

Let us first give an outline of the proof. The idea is to construct from a given processing situation, a Debreu-type of exchange economy with land (Debreu (1959), see also Debreu (1983)) and find a price equilibrium. Similarly as in Chapter 3, we derive a TU-game from this economy. This TU-game will be proven to be equivalent to the cost-game of the processing situation and as a result, the price equilibrium of the economy can be converted into a core allocation of the processing game. A similar technique has been used in Klijn, Tijs and Hamers (2000) to construct core elements of permutation games. Let us make this more formal.

An (initially) empty agenda is given. It will be a two-dimensional commodity. Of course time is one dimension, the other one is effort per time unit. In principle, there is no time restriction. The amount of effort per time unit is bounded by the capacity $\beta(N)$ of the grand coalition. At each moment of time, each player can buy any (measurable) part of the capacity available. Because of the two dimensions, it is customary to speak of land rather than of an agenda. Therefore, we are in fact dealing with exchange economies with land (see e.g., Legut,
Potters and Tijs (1994)). In order to complete their jobs, players must make reservations in the agenda. Only if a player books a block of time and effort per time unit sufficiently large to process his job, it will be completed. A price is chosen such that the market clears, i.e., no part of the agenda is booked by more than one player. This gives rise to a feasible schedule \( F \in \mathcal{F} \). Players are endowed with a part of the agenda proportionally to their capacities. Each player receives the revenues of his endowment. Clearing the market will lead to a price equilibrium. This Walrasian equilibrium can be converted into a core element of the processing game which will end the proof. The core allocation is stated right after Theorem 6.4. We have not found a direct proof (by plugging it into the core inequalities).

Let \( \mathcal{P} \) be a processing situation. Throughout this section we assume, without loss of generality, that \( N = \{1, \ldots, n\} \) such that \( \frac{p_1}{P_1} \geq \cdots \geq \frac{p_n}{P_n} \). Consider the exchange economy with land \( \mathcal{E}(\mathcal{P}) := (N, \{L, \mathcal{B}, \lambda\}, (A_i, V_i)_{i \in N}) \) in which:

- A commodity modeled by a measured space \( \{L, \mathcal{B}, \lambda\} \) has to be reallocated among the group of players \( N \). Here, \( L := [0, \beta(N)] \times \mathbb{R}_+ \) denotes a piece of land, \( \mathcal{B} \) is the Borel-\( \sigma \)-algebra of \( L \) and \( \lambda : \mathcal{B} \rightarrow \mathbb{R}_+ \) denotes the Lebesgue-measure on \( L \).
- Each player \( i \in N \) has an initial endowment \( A_i := \beta_i \times \mathbb{R}_+ \) in which \( \beta_i \) denotes the interval \( [\sum_{k<i} \beta_i, \sum_{k \leq i} \beta_i] \). Observe that \( \bigcup_{i \in N} A_i = L \) and \( \lambda(A_i \cap A_k) = 0 \) whenever \( i \neq k \).
- Each player \( i \in N \) has a reservation value \( V_i(B) \) for all sets \( B \in \mathcal{B} \) defined by
  \[
  V_i(B) := -\alpha_i \cdot T_i(B).
  \]
Here, \( T_i(B) := \inf\{t \in \mathbb{R}_+ | \int_0^t \int_0^{\beta_i} 1_B(x, \tau) \, dx \, d\tau \geq p_i\} \) is the indicator-function of set \( B \in \mathcal{B} \) (i.e., \( 1_B(x, \tau) = 1 \) if and only if \( (x, \tau) \in B \)).
So, \( T_i(B) \) denotes the moment of time at which the job of player \( i \) will be finished (the completion-time) in the case part \( B \) of the land (or agenda) is booked to work on his job.
In case subset \( B \) is not sufficient, \( T_i(B) \) equals infinity.
- Player \( i \in N \) has a quasi-linear utility function \( U_i : \mathcal{B} \times \mathbb{R} \rightarrow \mathbb{R} \) that denotes his valuation for bundles consisting of (pieces) of land and an amount of money. It is defined by
  \[
  U_i(B, x) := V_i(B) + x \quad \text{for all } B \in \mathcal{B} \text{ and } x \in \mathbb{R}.
  \]
Similarly, as in Section 3.2, we define an \( S \)-reallocation, a (strong) improvement and we associate a TU-game with the exchange economy \( \mathcal{E}(\mathcal{P}) \).

Let \( S \subseteq N \). An \( S \)-redistribution is a list \( (B_i)_{i \in S} \) of \( \lambda \)-measurable subsets of \( L \) with
\[
\bigcup_{i \in S} B_i \subseteq \bigcup_{i \in S} A_i \text{ and } \lambda(B_i \cap B_k) = 0 \text{ whenever } i \neq k.
\]
Let \( (B_i)_{i \in S} \) be an \( S \)-redistribution and \( x \in \mathbb{R}^S \) a vector such that \( \sum_{i \in S} x_i = 0 \). Then \( (B_i, x_i)_{i \in S} \) is an \( S \)-reallocation.

Let \( (B_i, x_i)_{i \in N} \) be an \( N \)-reallocation and let \( S \subseteq N \) be non-empty. Then an \( S \)-reallocation \( (C_i, y_i)_{i \in S} \) is an improvement upon \( (B_i, x_i)_{i \in N} \) if \( V_i(C_i) + y_i > V_i(B_i) + x_i \) for all \( i \in S \).

An \( N \)-reallocation \( (B_i, x_i)_{i \in N} \) is a core reallocation if no coalition \( S \) has an improvement upon \( (B_i, x_i)_{i \in N} \).

Given an exchange economy with land \( \mathcal{E}(\mathcal{P}) \) we define the TU-game \( (N, v_{\mathcal{E}(\mathcal{P})}) \) by
The TU-game \(\langle N, v_{E(P)}\rangle\) is in fact the TU-game \(\langle N, -c_P\rangle\), as the following lemma demonstrates.

**Lemma 6.7.** \(v_{E(P)}(S) = -c_P(S)\) for all \(S \subseteq N\).

**Proof.** Take \(S \subseteq N\), say \(S := \{i(1), \ldots, i(s)\}\) with \(i(1) < \ldots < i(s)\). Define for all \(k\) with \(1 \leq k \leq s\),

\[
B_k := \bigcup_{v \in S} [\beta_i] \times \frac{1}{\mu(S)} [p_i(k)],
\]

in which \([p_i(k)]\) denotes the interval \([\sum_{\ell < k} p_i(\ell), \sum_{\ell \leq k} p_i(\ell)]\) with length \(p_i(k)\). Clearly, \((B_k)_{1 \leq k \leq s}\) is an \(S\)-redistribution. It can be interpreted as a feasible schedule for the players in \(S\) to complete their jobs, i.e., during the interval \([p_i(k)]\) the capacity \(\beta(S)\) is used to complete the job of player \(i(k)\). In fact, we have,

\[
v_{E(P)}(S) \geq \sum_{k=1}^{s} V_i(k)(B_k) = -\sum_{k=1}^{s} \alpha_i(k) \cdot T_i(k)(B_k) = -\sum_{k=1}^{s} \frac{\alpha_i(k)}{\mu(S)} (p_i(1) + \ldots + p_i(k)) = -c_P(S).
\]

So, \(c_P(S) \geq -v_{E(P)}(S)\). However, due to the optimality of \(c_P(S)\), the converse of this inequality also holds. Hence, \(c_P(S) = -v_{E(P)}(S)\). \(\square\)

Next, we recall the concept of a price equilibrium (see also Section 3.4).

**Definition.** An \(N\)-reallocation \((B_i, x_i)_{i \in N}\) is a price equilibrium if there exists a price density function, i.e., a measurable function \(\pi : L \rightarrow \mathbb{R} (\pi \neq 0)\) such that

(i) \(P_\pi(B_i) + x_i = P_\pi(A_i)\) for all \(i \in N\),

(ii) If \(V_i(C) + y > V_i(B_i) + x_i\) for a certain \(C \subseteq L\), \(y \in \mathbb{R}\) and \(i \in N\),

then \(P_\pi(C) + y > P_\pi(A_i)\),

in which \(P_\pi(B) := \int_0^{\infty} \int_0^{\beta(N)} 1_B(x, t) \cdot \pi(x, t) \, dx \, dt\) for all \(B \subseteq L\).

In this definition of a price equilibrium, the price density function can be understood as the map which assigns to each point \((x, t)\) in \(L\) its price. Therefore, the price of a piece of land \(B \subseteq L\) equals the total price of all points contained in this land \(B\) which results in the definition of \(P_\pi(B)\) above.

The following lemma gives us a relation between the existence of a price equilibrium in the exchange economy \(E(P)\) and the non-emptiness of the core of the TU-game \(\langle N, -c_P\rangle\).

**Lemma 6.8.** If \((B_i, x_i)_{i \in N}\) is a price equilibrium in \(E(P)\), then \((V_i(B_i) + x_i)_{i \in N} \in \mathbb{R}^N\) is a core allocation of the TU-game \(\langle N, -c_P\rangle\).
Proof. Let \((B_i, x_i)_{i \in N}\) be a price equilibrium supported by the price density function \(\pi : L \rightarrow \mathbb{R}\). We prove that the vector \((V_i(B_i) + x_i)_{i \in N}\) is contained in the core \(C(-c_P)\) of the TU-game \(\langle N, -c_P \rangle\).

Suppose there exists a coalition \(S \subseteq N\) such that
\[
\sum_{i \in S} (V_i(B_i) + x_i) < -c_P(S).
\]

Let \((C_i)_{i \in S}\) be an \(S\)-redistribution such that \(\nu_E(P)(S) = \sum_{i \in S} V_i(C_i)\). Due to Lemma 6.7, we have that \(-c_P(S) = \sum_{i \in S} V_i(C_i)\). Define \(\varepsilon := \frac{1}{|S|} \sum_{i \in S} (V_i(C_i) - V_i(B_i) - x_i)\), then \(\varepsilon > 0\). Furthermore, define for all \(i \in S\),
\[
y_i := V_i(B_i) + x_i - V_i(C_i) + \varepsilon.
\]

Then the \(S\)-reallocation \((C_i, y_i)_{i \in S}\) is an improvement upon \((B_i, x_i)_{i \in N}\). Therefore, according to the maximality conditions, this yields that
\[
P_e(C_i) + y_i > P_e(A_i) \text{ for all } i \in S.
\]

Taking the sum over all \(i \in S\) yields the desired contradiction. \(\square\)

So, the existence of a price equilibrium in \(E(P)\) implies balancedness of the TU-game \(\langle N, -c_P \rangle\) and thus also balancedness of the cost-game \(\langle N, c_P \rangle\). Therefore, the proof of Theorem 6.4 boils down to the following proposition.

**Proposition 6.9.** The exchange economy with land \(E(P)\) has a price equilibrium.

Proof. Denote \([p_i]\) as the interval \([\sum_{k<i} p_k, \sum_{k\leq i} p_k]\) with length \(p_i\) for all \(i \in N\) and define
\[
\mu(t) := \begin{cases} 
\frac{p_i}{p_i} & \text{if } t \in \left[\frac{1}{N}, \frac{1}{N}\right] \sum_{k\leq i} p_k.
\end{cases}
\]

Observe that \(\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) is weakly decreasing. Let \((B_i)_{i \in N}\) be the \(N\)-redistribution defined by
\[
B_i := [0, \beta(N)] \times \frac{1}{\pi_N} [p_i] \text{ for all } i \in N.
\]

Furthermore, we define the price density function \(\pi : L \rightarrow \mathbb{R}\) by
\[
\pi(x, t) := \int_t^\infty \mu(\tau) d\tau.
\]

We prove that the \(N\)-redistribution \((B_i)_{i \in N}\) can be extended to a price equilibrium supported by this price density function. To do so, we first need to calculate the prices of \(A_i\) and \(B_i\), respectively.

Let us first calculate for every \(i \in N\) the price of \(B_i\). Take \(i \in N\), then
\[
P_\pi(B_i) = \int_0^\infty \int_0^\beta(N) 1_{B_i}(x, t) \cdot \pi(x, t) \, dx \, dt = \int_{[\beta_i]} \int_0^\beta(N) \pi(x, t) \, dx \, dt
\]
\[
= \int_{[\beta_i]} \int_t^\infty \mu(\tau) \, d\tau \, dt = \beta(N) \int_{[\beta_i]} \int_{t}^\infty \mu(\tau) \, d\tau \, dt.
\]

Note that
\[
\int_{[\beta_i]} \int_t^\infty \mu(\tau) \, d\tau \, dt = \int_{[\beta_i]} \left( \int_t^\beta(N) \frac{1}{\beta_i} \sum_{k \leq i} p_k \, d\tau \right) \, dt + \sum_{k=i+1}^{n} \frac{p_k}{\beta(N)} \int_{[\beta_i]} \int_t^\infty \mu(\tau) \, d\tau \, dt
\]
\[
= \left( -\frac{1}{2} \frac{\alpha_i}{\beta_i} \left( \frac{1}{\beta(N)} \sum_{k \leq i} p_k - t \right)^2 \right)_{t \in [\beta_i]} + \frac{p_i}{\beta(N)} \sum_{k=i+1}^{n} \frac{p_k}{\beta(N)}
\]
\[
= \frac{p_i}{\beta(N)} \left( \sum_{k=i}^{n} \frac{\alpha_k}{\beta(N)} \right)^2 + \frac{p_i}{\beta(N)} \left( \alpha_i + \ldots + \alpha_n \right)
\]
\[
= \frac{p_i}{\beta(N)} \left( \frac{1}{2} \alpha_i + \alpha_{i+1} + \ldots + \alpha_n \right).
\]

Hence, from equation (6) it follows that for all \( i \in N \),
\[
P_\pi(B_i) = \beta(N) \int_{[\beta_i]} \int_t^\infty \mu(\tau) \, d\tau \, dt = \frac{p_i}{\beta(N)} \left( \frac{1}{2} \alpha_i + \alpha_{i+1} + \ldots + \alpha_n \right) = T_i.
\]

So, for every \( i \in N \) the price of \( B_i \) equals the tax deposit \( T_i \). Similarly, it can be derived that for every \( i \in N \) the price of \( A_i \) equals the subsidy \( S_i \) of player \( i \). Indeed,
\[
P_\pi(A_i) = \int_0^\infty \int_0^{\beta(N)} 1_{A_i}(x, t) \cdot \pi(x, t) \, dx \, dt
\]
\[
= \int_0^\infty \int_{[\beta_i]} \pi(x, t) \, dx \, dt
\]
\[
= \beta_i \int_0^\infty \int_t^\infty \mu(\tau) \, d\tau \, dt
\]
\[
= \beta_i \sum_{k \in N} \int_{[\beta_i]} \int_{t}^\infty \mu(\tau) \, d\tau \, dt
\]
\[
= \beta_i \sum_{k \in N} \frac{p_k}{\beta(N)} \left( \frac{1}{2} \alpha_k + \alpha_{k+1} + \ldots + \alpha_n \right)
\]
\[
= \frac{p_i}{\beta(N)} \sum_{k \in N} T_k.
\]

The fifth equality can be derived exactly along the lines of equation (6). Define for all \( i \in N \),
\[
x_i := P_\pi(A_i) - P_\pi(B_i) = \frac{\partial_i}{\beta_i} \sum_{k \in N} T_k - T_i.
\]
Then \((B_i, x_i)_{i \in N}\) is an \(N\)-reallocation which clearly satisfies the budget constraints with respect to the price density \(\pi : L \rightarrow \mathbb{R}\). Next, we prove that the maximality conditions are also satisfied. To obtain a contradiction, suppose there exists \(C \subseteq L, y \in \mathbb{R}\) and \(i \in N\) such that

\[
V_i(C) + y > V_i(B_i) + x_i \quad \text{and} \quad P_\pi(C) + y \leq P_\pi(A_i).
\]

(7)

The reservation value \(V_i(C)\) is, by definition, finite only if there exists a number \(t > 0\) such that \(\int_0^t \int_0^{\beta(N)} l(x, \tau) \, dx \, d\tau \geq p_i\). Let \(t > 0\) such that \(V_i(C) = -\alpha_i \cdot t\). Since the price density function \(\pi : (x, s) \mapsto \pi(x, s)\) does not depend on \(x\) and is decreasing in \(s\), we may assume without loss of generality that \(C \subseteq [0, \beta(N)] \times [t - \frac{s}{\beta(N)}, t]\). Indeed, the \(\lambda\)-measurable set \(C\) is the cheapest piece of land with reservation value \(-\alpha_i \cdot t\). In this case, because \(P_\pi(A_i) = P_\pi(B_i) + x_i\), the two inequalities in (7) yield,

\[
V_i(C_i) - P_\pi(C_i) \geq V_i(C_i) + y - P_\pi(A_i) \\
= V_i(C_i) + y - P_\pi(B_i) - x_i \\
> V_i(B_i) + x_i - P_\pi(B_i) - x_i = V_i(B_i) - P_\pi(B_i).
\]

Hence,

\[
V_i(C_i) - P_\pi(C_i) > V_i(B_i) - P_\pi(B_i).
\]

(8)

Next, we prove that inequality (8) cannot hold. To do so, we define the function \(f : \mathbb{R}_+ \rightarrow \mathbb{R}\) by

\[
f(t) := V_i(C_t) - P_\pi(C_t) = -\alpha_i \cdot t - \beta(N) \int_t^{\infty} \int_{\frac{s}{\beta(N)}}^{\infty} \mu(\zeta) \, d\zeta \, d\tau
\]

and compute its maximum value. Observe that \(f\) is differentiable on \(\mathbb{R}_+\) and

\[
f'(t) = -\alpha_i - \beta(N) \cdot \left(\int_{\frac{t}{\beta(N)}}^{\infty} \mu(\zeta) d\zeta - \int_{t}^{\infty} \mu(\zeta) d\zeta\right) = -\alpha_i + \beta(N) \int_t^{\infty} \frac{\mu(\zeta)}{\beta(N)} d\zeta.
\]

Hence, \(f'\) is also differentiable on \(\mathbb{R}_+\) and \(f''(t) = \beta(N) \cdot \left(\mu(t) - \mu(t - \frac{t}{\beta(N)})\right) \leq 0\). This inequality follows from the fact that \(\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) is weakly decreasing. So, \(f\) is a concave function. Therefore, its maximal value is taken in \(t\) whenever \(f'(t) = 0\). If \(t := \frac{1}{\beta(N)} \sum_{k \leq i} p_k\), then

\[
f'(t) = -\alpha_i + \beta(N) \int_{\frac{t}{\beta(N)}}^{\infty} \mu(\zeta) d\zeta = -\alpha_i + \beta(N) \cdot \frac{\mu(t)}{\beta(N)} \cdot \frac{\alpha_i}{p_i} = 0.
\]

Hence, \(f\) takes its maximal value in the point \(t = \frac{1}{\beta(N)} \sum_{k \leq i} p_k\) and thus

\[
f(t) \leq f\left(\frac{1}{\beta(N)} \sum_{k \leq i} p_k\right) = V_i(B_i) - P_\pi(B_i) \quad \text{for all} \quad t \in \mathbb{R}_+.
\]

This contradicts inequality (8) and thus we conclude that the \(N\)-reallocation \((B_i, x_i)_{i \in N}\) is a price equilibrium supported by the price density function \(\pi\).
The proof of Theorem 6.4 is now straightforward.

**Proof of Theorem 6.4.** Because the exchange economy with land \( E(P) \) has a price equilibrium, the TU-game \((N, -c_P)\) has according to Lemma 6.8 a non-empty core. This means there exists a vector \( y \in \mathbb{R}^N \) such that \( y(N) = -c_P(N) \) and \( y(S) \geq -c_P(S) \) for all \( S \subseteq N \). Equivalently, there exists a vector \( y' \in \mathbb{R}^N \) such that \( y'(N) = c_P(N) \) and \( y'(S) \leq c_P(S) \) for all \( S \subseteq N \). Hence, the cost-game \((N, c_P)\) is balanced.

The reason for the cost-game \((N, c_P)\) to be totally balanced, is the fact that the processing game restricted to a coalition \( S \subseteq N \) is again a processing game and thus balanced.

According to Lemma 6.8, the vector \((-x_i + V_i(B_i)) \in \mathbb{N} \) in which \( B_i \in L \) and \( x_i \in \mathbb{R} \) are defined for all \( i \in N \) as in the proof of Proposition 6.9, is a core allocation in the cost-game \((N, c_P)\). Elaborating this expression provides the core allocation \( X \in C(c_P) \) stated below Theorem 6.4.

### 6.5 Final remarks

We conclude this chapter with the following three remarks.

**Remark.** In the previous section it is illustrated that the proposed core allocation \( X \), stated below Theorem 6.4, is derived from a price equilibrium. In Section 3.4 it is shown that a price equilibrium is not a game-theoretical solution concept. This seems to imply that our proposed core allocation \( X \), stated below Theorem 6.4, is also not a game-theoretical solution, i.e., it is a solution for processing situations, not for processing games. This is indeed the case, as we now demonstrate.

Let \( N := \{1, 2\} \) and \( J := \{j_1, j_2\} \). We construct two processing situations \( P \) and \( P' \), respectively. The first one is defined by \((N, J, p, \alpha, \beta)\) with \( p := (2, 16) \), \( \alpha := (24, 8) \) and \( \beta := (1, 1) \) and the second one is defined as \((N, J, p', \alpha', \beta')\) with \( p' := (4, 8) \), \( \alpha' := (24, 16) \) and \( \beta' := (2, 1) \). It is not difficult to verify that both processing situations yield the same cost-game \((N, c)\) given by:

\[
c(N) = \frac{1}{2} \cdot (24 \cdot 2 + 8 \cdot 18) = \frac{1}{4} \cdot (24 \cdot 4 + 16 \cdot 12) = 96,
\]
\[
c(\{1\}) = 48 \text{ and } c(\{2\}) = 128.
\]

However, the contribution to the total costs by the players according to the core allocation, stated just after Theorem 6.4, is different for both processing situations. For \( P \) this cost-allocation equals

\[
X = (24 + 20 - 26, 72 + 32 - 26) = (18, 78)
\]

and for \( P' \) it equals

\[
X' = (32 + 37 \frac{1}{2} - 39 \frac{1}{2}, 64 + 21 \frac{1}{2} - 19 \frac{5}{2}) = (30 \frac{3}{2}, 65 \frac{5}{2}).
\]

Hence, the proposed core allocation below Theorem 6.4 is *not* a game-theoretical solution. However, it still has the appealing property that it is independent on which optimal order is
chosen to process the jobs (Proposition 6.5). A property which is clearly satisfied by every

game-theoretical solution concept, since the costs of a coalition is of course the same in any

optimal order.

Next, we look at two classes of cost-games which can be proven to be totally balanced using

Theorem 6.4.

**Remark.** The following two classes of cost-games are totally balanced:

(i) Let \( N \) be a set of players, with \(|N| := n\), and let \( \beta \in \mathbb{R}^N_+ \). Then the cost-game \( \langle N, c \rangle \) defined by

\[
c(S) := \frac{1}{\beta(S)} |S| \cdot (|S| + 1)
\]

is totally balanced. This statement follows from Theorem 6.4 combined with the fact that

the processing game \( \langle N, c_P \rangle \) derived from the processing situation \( P := \langle N, J, p, \alpha, \beta \rangle \) with

\( p_j = 1 \) and \( \alpha_j = 1 \) for all \( j \in J \) is, according to Proposition 6.2, given by

\[
c_P(S) := \frac{1}{P} c(S)
\]

for all coalitions \( S \subseteq N \). As a result, the allocation \( X \in \mathbb{R}^N \) defined by

\[
X_i := \frac{1}{\beta(N)} (2 \cdot n + 1 - \beta_i \cdot \frac{n^2}{\beta(N)}) \text{ for all } i \in N
\]

is contained in the core \( C(c) \).

(ii) Let \( N := \{1, \ldots, n\} \) be a set of players and \( \alpha \in \mathbb{R}^N_+ \) be a vector such that \( \alpha_1 \geq \ldots \geq \alpha_n \). Then the cost-game \( \langle N, c \rangle \) defined by

\[
c(S) := \frac{1}{n} \cdot (1 \cdot \alpha_{i(1)} + 2 \cdot \alpha_{i(2)} + \ldots + s \cdot \alpha_{i(s)}) \text{ whenever } S = \{i(1), \ldots, i(s)\}
\]

is totally balanced. This statement follows again from Theorem 6.4 in combination with the

fact that, according to Proposition 6.2, this cost-game is the processing game \( \langle N, c_P \rangle \) derived from the processing situation \( P := \langle N, J, p, \alpha, \beta \rangle \) with \( p_i = 1 \) and \( \beta_i = 1 \) for all \( i \in N \). As a result, the allocation \( X \in \mathbb{R}^N \) defined by

\[
X_i := \frac{1}{n} \cdot \alpha_i \cdot \frac{i(i+1)}{2} + \frac{1}{n} \left( \frac{1}{n} \cdot \alpha_i + \sum_{k=i+1}^{n} \alpha_k \right) - \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{n} \cdot \alpha_i + \sum_{k=i+1}^{n} \alpha_k \right) \text{ for all } i \in N
\]

is contained in the core \( C(c) \).

The final remark discusses a generalization of a processing situation.

**Remark.** Initially, in a processing situation we assume that each player has to complete

exactly one job. In Quant et al. (2004) a generalization is studied in which each player,

again endowed with an individual capacity, has to complete several jobs. Again, each job

requires a certain processing demand. Moreover, each player now may have interest in the

completion of more than one job and the degrees of interest may vary among the agents.

More particularly, the cost-coefficient does not only depend on the jobs, but also on the

players. Formally, a processing situation with shared interest can be described by a tuple

\( \langle N, J, (p_{ij})_{j \in J}, (\alpha_{ij})_{i \in N, j \in J}, (\beta_i)_{i \in N} \rangle \).
Again $N$ is a finite set of players and $J$ is a finite set of jobs. The cardinalities of $N$ and $J$ may differ. Player $i \in N$ is endowed with a capacity of $\beta_i > 0$ to complete jobs. Job $j \in J$ has processing demand $p_j \in \mathbb{R}_+$ and during the time job $j$ is uncompleted, it generates a cost $\alpha_{ij} \in \mathbb{R}_+$ per time unit for player $i \in N$. So, the cost-coefficient does not only depend on the job, but also on the player. Analogous, to the original setting, a coalition $S \subseteq N$ of cooperating players faces a processing problem. In this case they face the processing problem

$$\langle J, (p_j)_{j \in J}, (\alpha_{Sj})_{j \in J}, \beta(S) \rangle,$$

in which $\alpha_{Sj} := \sum_{i \in S} \alpha_{ij}$ for all $j \in J$ and $\beta(S) := \sum_{i \in S} \beta_i$. As a result, one can construct a cooperative cost-game, in which the costs for a coalition equal the minimal costs of its corresponding processing problem. These cost-games are also totally balanced.

Let $\alpha_{Nj} := \sum_{i \in N} \alpha_{ij}$ for all $j \in J$ and assume that the jobs in $J := \{1, \ldots, m\}$ are ordered such that

$$\frac{\alpha_{N1}}{p_1} \geq \cdots \geq \frac{\alpha_{Nm}}{p_m}.$$

Furthermore, define for each job $j \in J$,

$$\pi_j := \frac{p_j}{\beta(\mathcal{N})} \cdot \left( \frac{1}{2} \alpha_{Nj} + \sum_{k=j+1}^{m} \alpha_{Nk} \right).$$

Then the vector $X \in \mathbb{R}^N$ given by

$$X_i := \sum_{j \in J} \alpha_{ij} \cdot \left( \sum_{k=1}^{j} \frac{p_k}{\beta(\mathcal{N})} + \frac{\pi_j}{\alpha_{Nj}} \right) - \frac{\beta_i}{\beta(\mathcal{N})} \sum_{j \in J} \pi_j \quad \text{for all } i \in N,$$

is a core allocation of the cost-game derived from the processing situation with shared interest. A proof of this statement which uses Theorem 6.4 can be found in Quant et al. (2004). So, also this class of cost-games is totally balanced. Furthermore, this core allocation turns out still to be independent from the optimal order chosen (if there are several). So, Corollary 6.6 remains valid for this core allocation.

Observe, that in case the cardinalities of $N$ and $J$ coincide and that for every $i \in N$ there exists exactly one $j \in J$ such that $\alpha_{ij} \neq 0$, we are dealing with a processing situation in the original setting. In this case, the described core element above coincides with the core element given in Section 6.3 for a processing game in the original setting.
Dynamic Selection in Normal-Form Games

7.1 Introduction

The final chapter of this thesis deals with Non-cooperative Game Theory. It investigates for $n$-person normal-form games the convergence to Nash-equilibria and stability of Nash-equilibria for a class of dynamic selection processes which are defined by a set of (ordinary) differential equations. Most of the results in the chapter can also be found in Meertens, Potters and Reijnierse (2006a).

The best-known and extensively used solution concept of non-cooperative game theory is the Nash-equilibrium (Nash (1950)). It is the benchmark theoretical solution for strategic interactions among a number of players in a non-cooperative environment. A Nash-equilibrium can be seen as an agreement which tells every player, involved in the conflict, exactly what to do, in such a way that none of the players has an incentive to deviate unilaterally from it. However, it does *not* give insight on how players come to play a Nash-equilibrium. Moreover, the multiplicity of Nash-equilibria gives rise to the question *which* equilibrium is more likely to be played if a non-cooperative game has more than one Nash-equilibrium. In an attempt to answer both questions researchers started to investigate alternative foundations of Nash-equilibria. One of these alternative foundations are evolutionary explanations. This approach was motivated by the work of biologists in evolutionary game theory (in particular, the work of Maynard Smith and Price (1973), see also Maynard Smith (1982)). Instead of asking whether players are rational in some well-defined sense, people ask whether an evolutionary selection process has a tendency towards a Nash-equilibrium. Moreover, if a selection process has such a tendency, the question ‘towards which Nash-equilibrium’ can be studied.

An approach to formalize the idea above is the following. Each player in the game is replaced by an uncountable large population. Within every population there are a finite number of
sets of individuals representing a certain pure strategy of the player, i.e., the individuals in such a set are ‘programmed’ to use the same pure strategy of the player whose role they play. Furthermore, the payoff to an individual is assumed to represent fitness, expressed by the number of offspring. So, more successful individuals get more offspring than less successful individuals. Because it is assumed that the offspring of an individual inherits its single parent’s strategy, the measure of each set of individuals, representing a certain pure strategy of the player, will change within each generation. This means that over time the relative measure of each set of individuals in every population, and thereby the strategy distribution in every population, changes according to some dynamic selection process.

In the literature several dynamic selection processes can be found. Let us recall some of them. One of the dynamics studied most extensively in evolutionary game theory is the so-called replicator dynamics (see e.g., Taylor (1979), Hofbauer and Sigmund (1988), Friedman (1991), Samuelson and Zhang (1992), Swinkels (1993), Ritzberger and Weibull (1995)). Other dynamic selection processes closely related to the replicator dynamics are aggregate monotonic dynamics (Samuelson and Zhang (1992)), sign-preserving dynamics (Ritzberger and Weibull (1995)) and monotonic dynamics (Ritzberger and Weibull (1995)). Each of these selection processes are defined by a set of (ordinary) differential equations on the polyhedron of mixed strategy profiles. They all have the following two properties in common:

(i) Extinct strategies stay extinct forever. If the set of individuals in a population playing a certain pure strategy has measure zero, it will keep measure zero in the next generations,

(ii) Every Nash-equilibrium is a rest-point of the set of differential equations. If the strategy distribution in each population is according to a Nash-equilibrium, this distribution will remain the same in the next generations. However, the converse is not true. This means that the dynamic selection process can get ‘stuck’ in a point which is not a Nash-equilibrium. In fact, for each of these dynamics every pure strategy profile is a rest-point.

More recently, dynamic selection processes are studied which are based on Nash’s proofs for the existence of Nash-equilibria (Nash (1950), Nash (1951)). An example of such a dynamic selection process is the so-called Brown-Von Neumann-Nash dynamics (see e.g., Berger and Hofbauer (2000), Hofbauer (2000), Sandholm (2001)). Its name is derived from the fact that this dynamic selection process is based on the iteration map used by Nash (1951) and from the fact that even earlier Brown and Von Neumann (1950) applied this dynamic selection process in the case of symmetric zero-sum games, for proving the existence of the value for such games. In this chapter we investigate a class of dynamics which include the Brown-Von Neumann-Nash dynamics. This class of dynamics is also defined by a set of (ordinary) differential equations on the polyhedron of mixed strategy profiles and since they are also based on Nash’s iteration map, they all have the attractive property that only Nash-equilibria are rest-points. This implies that if these dynamics converge, the limit is a Nash-equilibrium. We define these dynamic selection processes by so-called regret-functions. These functions measure the ‘regret’ of a player of not having played a certain pure strategy. By letting the measure of a set of individuals, representing a certain pure strategy of a player, grow over time according to some positive function of its regret, we obtain the dynamics investigated in this chapter. As the regret of a pure strategy can be positive even if the weight on this pure strategy is equal to zero, extinct strategies do not have to stay extinct forever. That is, even if the set of
individuals in a population playing a certain pure strategy has measure zero, it is possible that in the next generation it has a positive measure. An interpretation could be that ‘mutations’ prosper in the offspring. On the other hand, if the regret of a pure strategy is positive, it does not imply that the weight on this pure strategy will increase. So, even if a certain pure strategy is successful in a generation, the relative measure of the set of individuals playing this pure strategy can become smaller within the next generation. The reason is that there are already too many individuals in the population playing this (successful) pure strategy. An interpretation could be that these individuals ‘hamper’ each other, i.e., they ‘suffer’ from their own success.

Our main goal is to investigate whether these dynamics converge towards the set of Nash-equilibria only, i.e., if every limit set is a subset of the set of Nash-equilibria. In general this is not the case. This follows from a recent result by Hart and Mas-Colell (2003). However, for (two-person) zero-sum games (and strategic equivalent games) every limit set of the dynamics studied in this chapter consists of Nash-equilibria only. Since, this class of dynamics includes the Brown-Von Neumann-Nash dynamics this result generalizes the classic result by Brown and Von Neumann (1950) to the asymmetric case. Furthermore, it implies that the (unique) Nash-component of a zero-sum game is asymptotically stable and that a zero-sum game has an asymptotically stable Nash-equilibrium if and only if it has a unique Nash-equilibrium. Furthermore, on the class of (n-person) potential games (Monderer and Shapley (1996)) every limit set of the dynamics is also a subset of the set of Nash-equilibria and the results of Sandholm (2001) tell us that only strict Nash-equilibria are asymptotically stable.

From the results about stability, it follows that many normal-form games do not possess an asymptotically stable Nash-equilibrium. Of course, this is due to the fact that an asymptotically stable strategy profile must be isolated in the space of rest-points. In fact, one can show that the smallest possible asymptotically stable set of Nash-equilibria is a connected component in the sense of Kohlberg and Mertens (1986). This problem is already pointed out by Swinkels (1993). Ritzberger and Weibull (1995) even conclude:

"Hence, the connection between evolutionary selection in n-player games and Nash equilibrium is weaker than it may first appear" (page 1372–1373).

Therefore, the attention in evolutionary selection dynamics shifted towards connected components of Nash-equilibria (see e.g., Ritzberger and Weibull (1995), Demichelis and Ritzberger (2000), Demichelis and Germano (2002)). Nevertheless, although such a connected component may be asymptotically stable it does not imply that every strategy profile within this component is contained in the limit set of some (starting) profile outside the component. Differently, the dynamics may select only a proper subset of this connected component, but the concept of asymptotic stability is not able to distinguish between this set and the connected component. Hence, it cannot precisely predict which Nash-equilibria will be selected by the dynamics. Therefore, it seems that the concept of asymptotic stability may be a too demanding stability concept for evolutionary selection dynamics. To emphasize this point, we give an example of a potential game for which the (unique) connected component of the set of Nash-equilibria is asymptotically stable, but for a class of dynamics every limit set of a strategy profile consists of one and the same Nash-equilibrium.

We start by giving the formal definition of a normal-form game and of the Nash-equilibrium.
7.2 Normal-form games

7.2.1 Preliminaries

Non-cooperative game theory has become a standard tool in modeling conflict situations between rational players. The basic model in non-cooperative game theory is the finite n-person game in normal-form.

Let $\Gamma$ be a finite n-person normal-form game with player set $N$. Each player $i \in N$ has a (non-empty) set $S_i$ consisting of $K_i$ pure strategies $s_i^k$, $1 \leq k \leq K_i$. We define $S := \prod_{i \in N} S_i$ as the set of pure strategy profiles $s = (s_1^1, \ldots, s_n^n)$. When playing the game, each player $i \in N$ chooses, without telling the other players, a pure strategy $s_i^k \in S_i$ to his disposal and by doing so, each player $i \in N$ receives a payoff (or utility) $U_i : S \rightarrow \mathbb{R}$. Furthermore, we allow that players randomize over their pure strategies, meaning that for each player $i \in N$ we extend the set of pure strategies $S_i$ to the set of mixed strategies $\Delta_i := \{\sigma_i \in \mathbb{R}^{K_i} \mid \sum_{k=1}^{K_i} \sigma_i^k = 1\}$, i.e., $\sigma_i^k$ is the weight player $i \in N$ puts on the pure strategy $s_i^k$. By identifying each pure strategy $s_i^k \in S_i$ with the corresponding unit vector $e_i^k \in \Delta_i$ for all $i \in N$, the set of pure strategies $S_i$ can be identified with the set of vertices of $\Delta_i$. We define $\Delta := \prod_{i \in N} \Delta_i$, as the polyhedron of mixed strategy profiles $\sigma = (\sigma_1, \ldots, \sigma_n)$. If $\sigma \in \Delta$ is a strategy profile, then $\sigma_{-i} \in \Delta_{-i}$ is the strategy profile $(\sigma_j)_{j \neq i}$. Furthermore, we extend the payoff function $U_i : S \rightarrow \mathbb{R}$ of player $i \in N$ to the multilinear (expected) payoff function $U_i : \Delta \rightarrow \mathbb{R}$ which is defined in the usual manner, i.e.,

$$U_i(\sigma) := \sum_{(s_1^1, \ldots, s_n^n) \in S} \left( \prod_{j \in N} \sigma_j^{k_j} \right) U_i(s_1^1, \ldots, s_n^n) \quad \text{for all } i \in N.$$

Hence, a normal-form game $\Gamma$ is characterized by the tuple $(N, (\Delta_i)_{i \in N}, (U_i)_{i \in N})$.

Because of the rationality of the players in the normal-form game $\Gamma$, a player $i \in N$ always tries to maximize his payoff $U_i : \Delta \rightarrow \mathbb{R}$, given the strategies of the other players. In other words, given the strategies of the other players, player $i$ tries to find a best reply to the strategies of his opponents. Let us introduce some notations in this context.

For player $i \in N$ a strategy $\sigma_i \in \Delta_i$ is a best reply of player $i$ to $\sigma_{-i} \in \Delta_{-i}$ if

$$U_i(\sigma_{-i}, \sigma_i) \geq U_i(\sigma_{-i}, \hat{\sigma}_i) \quad \text{for all } \hat{\sigma}_i \in \Delta_i.$$

The set of best replies of player $i \in N$ to $\sigma_{-i} \in \Delta_{-i}$ is denoted by $\beta_i(\sigma_{-i})$. The correspondence $\beta : \Delta \rightarrow \Delta$ for all $\sigma \in \Delta$ defined by

$$\beta(\sigma) := \prod_{i \in N} \beta_i(\sigma_{-i})$$

is the best reply correspondence of $\Gamma$. For player $i \in N$ a pure strategy $s_i^k \in S_i$ is a pure best reply of player $i$ to $\sigma_{-i} \in \Delta_{-i}$ if $s_i^k \in \beta_i(\sigma_{-i})$. The set of all pure best replies of player $i \in N$ to $\sigma_{-i} \in \Delta_{-i}$ is denoted by $PB_i(\sigma_{-i})$. For every strategy profile $\sigma \in \Delta$ we define $PB(\sigma) := \prod_{i \in N} PB_i(\sigma_{-i})$ as the set of pure best replies to $\sigma$. 


7.2 Normal-form games

For a player \(i \in N\) and strategy \(\sigma_i \in \Delta_i\), the carrier of \(\sigma_i\) is defined by

\[
C(\sigma_i) := \{s_i^k \in S_i \mid s_i^k > 0\}.
\]

The carrier of \(\sigma \in \Delta\) is defined as

\[
C(\sigma) := \prod_{i=1}^{n} C(\sigma_i).
\]

7.2.2 The Nash-equilibrium

Probably the most famous and extensively studied solution concept for normal-form games is introduced by Nash (1950).

**Definition (Nash (1950)).** Let \(\Gamma := (N, (\Delta_i)_{i \in N}, (U_i)_{i \in N})\) be a normal-form game. A strategy profile \(\sigma \in \Delta\) is a Nash-equilibrium, if for all \(i \in N\),

\[
U_i(\sigma_{-i}, \sigma_i) \geq U_i(\sigma_{-i}, \sigma_i^*) \quad \text{for every } \sigma_i^* \in \Delta_i.
\]

We denote \(NE(\Gamma)\) as the set of Nash-equilibria of \(\Gamma\).

So, a Nash-equilibrium is an equilibrium in the sense that it does not allow for an unilateral improvement by any of the players. Nash (1951) proves that normal-form games have Nash-equilibria. To be complete, we also give a proof for the existence of Nash-equilibria in normal-form games. To do so, we define so-called regret-functions of pure strategies. These functions measure the ‘regret’ of a player of not having played a certain pure strategy, expressed in the player’s payoff.

**Definition.** Let \(\sigma \in \Delta\) be a strategy profile. For all \(1 \leq k \leq K_i\) and \(i \in N\) we define the regret-function of pure strategy \(s_i^k \in S_i\) by

\[
R_i^k(\sigma) := [U_i(\sigma_{-i}, s_i^k) - U_i(\sigma_{-i}, \sigma_i)]_+.
\]

Recall that \([a]_+\) denotes \(\max\{a, 0\}\).

Via the regret-functions and the fixed-point theorem of Brouwer (Brouwer (1912)) one can prove the existence of Nash-equilibria in normal-form games.

**Theorem 7.1 (Nash (1951)).** For every normal-form game \(\Gamma\) the set of Nash-equilibria \(NE(\Gamma)\) is non-empty.

**Proof.** Let \(\Gamma\) be a normal-form game and take \(\sigma \in \Delta\). Define for all \(1 \leq k \leq K_i\) and \(i \in N\),

\[
R_i^k(\sigma) := \frac{R_i^k(\sigma) + \sigma_i^k}{\sum_{r=1}^{K_i} R_i^r(\sigma) + 1}.
\]

Note that \(R_i^k(\sigma) \geq 0\) for all \(1 \leq k \leq K_i\) and \(i \in N\) and note that \(\sum_{k=1}^{K_i} R_i^k(\sigma) = 1\) for all \(i \in N\). Define the map \(R : \Delta \rightarrow \Delta\) by

\[
\sigma \mapsto \left( (R_i^k(\sigma))_{1 \leq k \leq K_i} \right)_{1 \leq i \leq n}.
\]

The map \(R\) is a continuous map of a compact and convex set into itself. Therefore, according to the theorem of Brouwer it has a fixed point. We prove that such a fixed point is a Nash-equilibrium.
Let $\sigma \in \Delta$ be a fixed point of the map $R$, i.e., $\sigma^k_i = R^k_i(\sigma)$ for all $1 \leq k \leq K_i$ and $i \in N$. Writing out this expression yields,

$$R^k_i(\sigma) = \sigma^k_i \sum_{\ell=1}^{K_i} R^\ell_i(\sigma) \text{ for all } 1 \leq k \leq K_i \text{ and } i \in N.$$ 

Suppose $\sum_{\ell=1}^{K_i} R^\ell_i(\sigma) > 0$ for some $i \in N$. Then

$$\sum_{k : s^k_i \in C(\sigma)} \sigma^k_i \cdot R^k_i(\sigma) = \sum_{k : s^k_i \in C(\sigma)} (\sigma^k_i)^2 \cdot \sum_{\ell=1}^{K_i} R^\ell_i(\sigma) > 0.$$ 

But, because $\sigma^k_i > 0 \iff R^k_i(\sigma) > 0$, we also have, by the multilinearity of $U_i$,

$$\sum_{k : s^k_i \in C(\sigma)} \sigma^k_i \cdot R^k_i(\sigma) = \sum_{k : s^k_i \in C(\sigma)} \sigma^k_i \cdot [U_i(\sigma_{-i}, s^k_i) - U_i(\sigma_{-i}, \sigma_i)] = 0.$$ 

Contradiction. Hence, $\sum_{\ell=1}^{K_i} R^\ell_i(\sigma) = 0$ for all $i \in N$ and therefore $R^k_i(\sigma) = 0$ for all $1 \leq k \leq K_i$ and $i \in N$. This means that $U_i(\sigma_{-i}, \sigma_i) \geq U_i(\sigma_{-i}, s^k_i)$ for all $1 \leq k \leq K_i$ and $i \in N$. Therefore, by multilinearity of the payoff functions, it follows that for all $i \in N$,

$$U_i(\sigma_{-i}, \sigma_i) \geq U_i(\sigma_{-i}, \sigma_i) \text{ for every } \sigma_i \in \Delta_i.$$ 

Hence, $\sigma$ is a Nash-equilibrium.

The proof of Theorem 7.1 demonstrates that in every fixed point $\sigma$ of the map $R : \Delta \to \Delta$ the regret $R^k_i(\sigma)$ of pure strategy $s^k_i$ is zero for all $1 \leq k \leq K_i$ and $i \in N$ which, in turn, implies that $\sigma$ is a Nash-equilibrium. Observe that the converse of this statement is also true. In fact, it is straightforward to verify the following result.

**Corollary 7.2.** Let $\Gamma$ be a normal-form game. The following statements are equivalent:

(i) $\sigma$ is a Nash-equilibrium of $\Gamma$,

(ii) $\sigma \in \beta(\sigma)$,

(iii) $C(\sigma) \subseteq PB(\sigma)$,

(iv) $\sigma$ is a fixed point of the map $R : \Delta \to \Delta$,

(v) $R^k_i(\sigma) = 0$ for all $1 \leq k \leq K_i$ and $i \in N$.

So, the set of Nash-equilibria of a normal-form game $\Gamma$ is non-empty. Kohlberg and Mertens (1986) prove that the set of Nash-equilibria $NE(\Gamma)$ is the union of finitely many disjoint, closed and connected sets, called connected components. Such a connected component $K$ of $NE(\Gamma)$ is maximal in the sense that there is no closed and connected subset of $NE(\Gamma)$ that properly contains $K$. 
7.3 Evolution and the Nash-equilibrium

After a Nash-equilibrium is played, it follows from the rationality of the players that no player has an incentive to deviate unilaterally from it. Indeed, if a player \( i \in N \) plays strategy \( \sigma_i \in \Delta, \) as his part of the Nash-equilibrium \( \sigma \in \Delta, \) then \( \sigma_i \) is a best reply to \( \sigma_{-i}. \) This property of Nash-equilibria is known as self-enforcingness. So, looking back, each player may be content with the Nash-equilibrium as an outcome, since a player cannot improve from it by himself. However, it does not answer the question:

(i) ‘How players come to play a Nash-equilibrium?’

In applying the concept of the Nash-equilibrium, game theorists became aware of a serious drawback of the Nash-equilibrium, namely, the multiplicity of Nash-equilibria in a given normal-form game and the inconsistency of many Nash-equilibria with the intuitive notion about what an equilibrium of the game should be. For instance, it is possible that in a Nash-equilibrium players use weakly dominated strategies which from a strategic point of view can be considered to be unreasonable. Therefore, these Nash-equilibria are not convincing as an outcome of the game. This observation was in the 1970’s the starting point of the theory of refinements, i.e., the theory to distinguish between several Nash-equilibria by putting more requirements on the original concept of this equilibrium. Several results have been obtained along this line of research and eventually it culminated into the work of Kohlberg and Mertens (1986), but not before several refinements had been introduced. Here we recall some of them, the ones which we use later on in this chapter. For a more complete overview of the refinements of Nash-equilibria, the reader is referred to Van Damme (1991).

The notion of a perfect Nash-equilibrium has been introduced by Selten in 1975 and is one of the most basic results in the theory of Nash-equilibrium refinements. A strategy profile \( \sigma \in \Delta \) is completely mixed if \( \sigma^k_i > 0 \) for all \( 1 \leq k \leq K_i, \) and \( i \in N. \)

**Definition (Selten (1975)).** A Nash-equilibrium \( \sigma \) is perfect if there exists a sequence of completely mixed strategy profiles \( \{\sigma_m\}_{m \in \mathbb{N}} \) such that

\[
\lim_{m \to \infty} \sigma_m = \sigma \quad \text{and} \quad \sigma \in \beta(\sigma_m) \quad \text{for all} \quad m \in \mathbb{N}
\]

From the definition it follows immediately that any Nash-equilibrium with completely mixed strategies is a perfect Nash-equilibrium. Indeed, if \( \sigma \) is completely mixed, one can choose \( \sigma_m = \sigma \) for all \( m \in \mathbb{N}. \) Furthermore, Selten (1975) proves that every normal-form game has at least one perfect Nash-equilibrium. An other more demanding refinement of the Nash-equilibrium is the concept of strictness.

**Definition.** A Nash-equilibrium \( \sigma \) is strict if \( \beta(\sigma) = \{\sigma\}, \) i.e., it is its own unique best reply. This means that

\[
U_i(\sigma_{-i}, \sigma_i) > U_i(\sigma_{-i}, \hat{\sigma}_i) \quad \text{for every} \quad \hat{\sigma}_i \in \Delta, \text{with} \quad \hat{\sigma}_i \neq \sigma_i \quad \text{and all} \quad i \in N.
\]

Observe that a strict Nash-equilibrium is, by continuity of the payoff function, the unique best reply to a neighborhood of itself and therefore, by multilinearity of the payoff function, strict
Nash-equilibria are pure strategy profiles. Furthermore, it is easy to verify that a strict Nash-equilibrium is in particular perfect. However, since a normal-form game does not always have pure Nash-equilibria, the existence of a strict Nash-equilibrium cannot be guaranteed.

Finally, we mention the concept of robustness which has been introduced by Okada (1983). A robust Nash-equilibrium is like the strict Nash-equilibrium also a best reply to a neighborhood of itself, however it may not be the only one.

**Definition (Okada (1983)).** A strategy profile \( \sigma \in \Delta \) is a robust Nash-equilibrium if there exists a number \( \varepsilon > 0 \) such that

\[
\sigma \in \beta(\tilde{\sigma}) \quad \text{for every} \quad \tilde{\sigma} \in B_\varepsilon(\sigma).
\]

The notation \( B_\varepsilon(\sigma) \) means \( \{ \tilde{\sigma} \in \Delta \mid |\tilde{\sigma} - \sigma| < \varepsilon \} \), where \( | \cdot | : \mathbb{R}^{[S]} \to \mathbb{R}_+ \) denotes the Euclidean metric.

Like strict Nash-equilibria, also robust Nash-equilibria are in particular perfect, but their existence cannot be guaranteed.

In the literature many more refinements of the Nash-equilibrium can be found and the theory of refinements has been proven helpful to eliminate inadequate Nash-equilibria. However, this theory has also its drawbacks. Not only the amount of refinements which have been developed, became rather large, and therefore it lost its transparency with respect to which refinement best suited which situation, but the theory of refinements also assumes that players act according to a high level of rationality. This may be too demanding. So, the following question remains:

(ii) ‘Which Nash-equilibrium is more likely to be played if a game has more than one Nash-equilibrium?’

In an attempt to answer the Questions (i) and (ii), game-theorists picked up the idea of evolutionary game theory, introduced by biologists in studying the evolution of populations of animals and the individual behavior of their members. The paper by Maynard Smith and Price (1973) is probably the most important in transferring evolutionary thinking into game theory. This paper directed game theorist’s attention away from their increasingly elaborate definitions of rationality. Because if evolutionary game theory can somehow predict the behavior of animals, rationality cannot be so crucial. So, instead of asking demanding notions of rationality, one may ask whether an evolutionary selection process between certain populations of animals converges towards Nash-equilibria. And if an evolutionary selection process converges one can study which Nash-equilibrium are in the limit set of such processes.

An approach to formalize this idea of evolutionary game theory is by replacing each player in the game by an uncountable large population of non-rational individuals. Within each of these populations there are a finite number of sets of individuals representing a certain pure strategy of the player, i.e., the individuals in such a set are ‘programmed’ to use the same pure strategy. So, a pure strategy is a behavioral policy which is not controlled by the individual itself, but it is controlled by his ‘instinct’. Furthermore, it is assumed that these behavior policies are heritable traits, i.e., if an individual is playing a certain pure strategy, then in the next generation its offspring will play the same pure strategy. Note that this implies asexual
reproduction, i.e., individuals have a single parent. Finally, the payoff to an individual is assumed to represent fitness, expressed by the number of offspring, i.e., within a generation individuals with a more successful behavior will have more offspring than individuals with a less successful behavior. So, the relative measure of each set of individuals, representing a certain pure strategy of the player, will grow or diminish in the next generation. This means that over time the relative measure of each set of individuals within every population, and by that the strategy distribution in every population, changes according to some dynamic selection process.

In the forthcoming section we describe the class of dynamic selection processes investigated in this chapter. This class of dynamic selection processes is defined by a set of (ordinary) differential equations on the polyhedron of mixed strategy profiles. These differential equations are defined by the regret-functions which we introduced in Subsection 7.2.2.

### 7.4 The dynamics

Let $f := \{f^k_i\}_{1 \leq k \leq K, i \in \mathcal{N}}$ be a collection of Lipschitz continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that for all $1 \leq k \leq K_i$ and $i \in \mathcal{N}$,

$$f^k_i(0) = 0 \text{ and } f^k_i(x) > 0 \text{ for all } x > 0.$$

We write $\mathcal{F}$ as the family of all such collections of functions $f$.

In Subsection 7.2.2, we introduced for all $1 \leq k \leq K_i$ and $i \in \mathcal{N}$ the regret-function $R^k_i : \Delta \rightarrow \mathbb{R}_+$ which measures the ‘regret’ of player $i \in \mathcal{N}$ of not having played pure strategy $s^k_i \in S_i$, expressed in the payoff $U_i$. With these regret-functions $R^k_i$ and a collection of functions $f \in \mathcal{F}$ we define a dynamic selection process with $n$ uncountable populations $i$ and $K_i$ types of individuals within every population $i$. Let $N^k_i(t)$ be the measure of the set of individuals of type $1 \leq k \leq K_i$ in population $i$ at time $t \geq 0$ and $N_i(t)$ the measure of the population $i$ at time $t \geq 0$. Furthermore, we define $h > 0$ as the step-size of the dynamic selection process. Given a time $t \geq 0$ and given the measures $N^k_i(t)$ and $N_i(t)$ of each type $1 \leq k \leq K_i$ and every population $i$, the set of individuals of type $k$ in a population $i$ has a constant growth-rate of $\rho^k_i(t)$ on the interval $[t, t+h)$, i.e., within one step. So, given the measures $N^k_i(t)$ and $N_i(t)$ of each type $1 \leq k \leq K_i$ and every population $i$ at a given time $t \geq 0$, the measure of the set of individuals of type $k$ in population $i$ in the next step, i.e., at time $t + h$ equals

$$N^k_i(t+h) = N^k_i(t) \cdot [1 + h \cdot \rho^k_i(t)].$$

The measure of population $i$ at time $t + h$ equals

$$N_i(t+h) = \sum_{k=1}^{K_i} N^k_i(t) \cdot [1 + h \cdot \rho^k_i(t)] = N_i(t) + h \cdot \sum_{k=1}^{K_i} N^k_i(t) \cdot \rho^k_i(t).$$

Introducing $\sigma^k_i(t) := N^k_i(t)/N_i(t)$ for each type $1 \leq k \leq K_i$ and every population $i$, yields,

$$\frac{1}{h} \cdot [\sigma^k_i(t+h) - \sigma^k_i(t)] = \frac{\sigma^k_i(t) \cdot \rho^k_i(t) - \sigma^k_i(t) \sum_{\ell=1}^{K_i} \sigma^\ell_i(t) \cdot \rho^\ell_i(t)}{1 + h \cdot \sum_{\ell=1}^{K_i} \sigma^\ell_i(t) \cdot \rho^\ell_i(t)}. $$
Proposition 7.3.

Proof. Let \( f \) be (\( D \)) and by letting the step-size \( h \) converge to zero we obtain the dynamics of our interest. Hence, the following set of differential equations (Df), defined on the polyhedron \( \Delta \), comports to these dynamics (time indices are suppressed):

\[
\frac{d}{dt}\sigma_i^k(t) = f_i^k \circ R_i^k(\sigma(t)) - \sigma_i^k \cdot f_i \circ R_i(\sigma) \quad \text{for all } 1 \leq k \leq K_i \text{ and } i \in N. \tag{Df}
\]

With abuse of notation we write \( f_i \circ R_i(\sigma) := \sum_{k=1}^{K_i} f_i^k \circ R_i^k(\sigma) \) for all \( i \in N \).

Remark.

(i) If \( f_i^k : x \mapsto x \) for all \( 1 \leq k \leq K_i \) and \( i \in N \), then (Df) yields the Brown-Von Neumann-Nash dynamics (see e.g., Berger and Hofbauer (2000), Hofbauer (2000), Sandholm (2001)).

(ii) For all \( 1 \leq k \leq K_i \) and \( i \in N \), \( \sigma_i^k = 0 \) does not imply \( \sigma_i^k = 0 \) and \( f_i^k \circ R_i^k(\sigma) > 0 \) (i.e., \( U_i(\sigma_{-i}, \sigma^k_{-i}) > U_i(\sigma_{-i}, \sigma_i) \)) does not imply \( \sigma_i^k > 0 \).

(iii) For all \( 1 \leq k \leq K_i \) and \( i \in N \), if \( \sigma_i^k = 0 \), then \( \sigma_i^k \geq 0 \) and furthermore, \( \sum_{k=1}^{K_i} \sigma_i^k = 0 \). Therefore,

\[
\sigma(t) \in \Delta \text{ for all } t \geq 0 \text{ whenever } \sigma(0) \in \Delta,
\]

(iv) For every \( f \in F \), the map \( \sigma \mapsto f_i^k \circ R_i^k(\sigma) - \sigma_i^k \cdot f_i \circ R_i(\sigma) \) is Lipschitz continuous on \( \Delta \) for all \( 1 \leq k \leq K_i \) and \( i \in N \).

(v) Let \( a \in \Delta \), then for every \( f \in F \) there is a unique solution \( \sigma_a : \mathbb{R}^+ \to \Delta \) of (Df) with \( \sigma_a(0) = a \) (Picard-Lindelöf Theorem, see e.g., Hirsch and Smale (1974)).

(vi) There exists a number \( L \geq 0 \) such that for every \( a, b \in \Delta \) and \( T > 0 \),

\[
|\sigma_a(T) - \sigma_b(T)| \leq e^{LT} |a - b|,
\]

(vii) If \( \sigma_a(T) := b \) for some \( T > 0 \), then \( \sigma_b(t) = \sigma_a(T + t) \) for all \( t \geq 0 \).

Let \( f \in F \). For every \( a \in \Delta \) we define \( O_a(f) := \{ \sigma_a(t) \mid t \geq 0 \} \) as the orbit of \( a \). A set \( A \subseteq \Delta \) is (\( Df \))-invariant, if \( O_a(f) \subseteq A \) for every \( a \in A \). By Remark (iii) the polyhedron \( \Delta \) is (\( Df \))-invariant and by Remark (vii) the orbit \( O_a(f) \) is (\( Df \))-invariant for every \( a \in \Delta \).

Next, we prove that the (relative) interior of \( \Delta \), i.e., \( \text{int}(\Delta) := \{ \sigma \in \Delta \mid \sigma_i^k > 0 \text{ for all } 1 \leq k \leq K_i \text{ and } i \in N \} \), is also (\( Df \))-invariant.

Proposition 7.3. \( \text{The interior of } \Delta \text{ is (Df)-invariant for every } f \in F. \)

Proof. Let \( f \in F \). Take \( a \in \text{int}(\Delta) \) and let \( \sigma(t) \) be the (unique) solution of (Df) with \( \sigma(0) = a \). Suppose there exist \( 1 \leq k \leq K_i, i \in N \) and a number \( T > 0 \) such that \( \sigma_i^k(T) = 0 \) and \( \sigma_i^k(t) > 0 \) for all \( t \in [0, T) \). Then for all \( t \in [0, T) \),

\[
\frac{d}{dt}\sigma_i^k(t) = f_i^k \circ R_i^k(\sigma(t)) - f_i \circ R_i(\sigma(t)) \geq f_i \circ R_i(\sigma(t)).
\]

Therefore,
7.5 The limit set and stability

$$\int_0^T \frac{d\sigma_k^i(t)}{\sigma_k^i(t)} \, dt \geq - \int_0^T f_i \circ R_i(\sigma(t)) \, dt.$$ 

Hence,

$$\lim_{t \to T} \log(\sigma_k^i(t)) - \log(\sigma_k^b(t)) \geq - \int_0^T f_i \circ R_i(\sigma(t)) \, dt.$$ 

But $\sigma_k^b(T) = 0$, $\sigma_k^i > 0$ and $\int_0^T f_i \circ R_i(\sigma(t)) \, dt < \infty$. Contradiction. \qed

For every $f \in \mathcal{F}$, a strategy profile $\sigma \in \Delta$ is a rest-point of $(Df)$, if $\{ \sigma \}$ is $(Df)$-invariant. Because the set of differential equations $(Df)$ is derived from the map $R : \Delta \to \Delta$ used in the proof of Theorem 7.1, it has the property that only Nash-equilibria are rest-points of $(Df)$ for every $f \in \mathcal{F}$. Intuitively, this means that every dynamics defined by $(Df)$ cannot get stuck in a strategy profile which is not a Nash-equilibrium. Let us make this more formal.

**Proposition 7.4.** Let $f \in \mathcal{F}$ and let $\sigma \in \Delta$ be a strategy profile. Then $\sigma$ is a rest-point of $(Df)$ if and only if $\sigma$ is a Nash-equilibrium.

**Proof.** $\Rightarrow$) Let $\sigma$ be a rest-point of $(Df)$. Then $f_i^k \circ R_i^k(\sigma) = \sigma_i^k \cdot f_i \circ R_i(\sigma)$ for all $1 \leq k \leq K_i$ and $i \in N$. Suppose $f_i \circ R_i(\sigma) > 0$ for some $i \in N$. Then

$$\sigma_i^k > 0 \Leftrightarrow f_i^k \circ R_i^k(\sigma) > 0 \Leftrightarrow R_i^k(\sigma) > 0.$$ 

Therefore,

$$\sum_{k : s_i^k \in C(\sigma_i)} \sigma_i^k \cdot R_i^k(\sigma) > 0.$$ 

But we also have that

$$\sum_{k : s_i^k \in C(\sigma_i)} \sigma_i^k \cdot R_i^k(\sigma) = \sum_{k : s_i^k \in C(\sigma_i)} \sigma_i^k \cdot [U_i(\sigma_{-i}, s_i^k) - U_i(\sigma_{-i}, \sigma_i)] = 0.$$ 

Contradiction. So, $f_i \circ R_i(\sigma) := \sum_{i=1}^{K_i} f_i^k \circ R_i^k(\sigma) = 0$ for all $i \in N$ and therefore by the definition of $f \in \mathcal{F}$, we have that $R_i^k(\sigma) = 0$ for all $1 \leq k \leq K_i$ and $i \in N$. According to Corollary 7.2(v) this means that $\sigma$ is a Nash-equilibrium.

$\Leftarrow$) Conversely, if $\sigma$ is a Nash-equilibrium, then $R_i^k(\sigma) = 0$ for all $1 \leq k \leq K_i$ and $i \in N$ (Corollary 7.2(v)). By definition of $f \in \mathcal{F}$, this yields that $\sigma_i^k = 0$ for all $1 \leq k \leq K_i$ and $i \in N$. \qed

### 7.5 The limit set and stability

This section contains the main results of our analysis. Our first result states that every strategy profile $b \in \Delta$ which is reachable under $(Df)$ (i.e., $\lim_{t \to \infty} \sigma_a(t) = b$ for some strategy profile $a \in \Delta$) is a Nash-equilibrium.

**Theorem 7.5.** Let $f \in \mathcal{F}$, if $b \in \Delta$ is reachable under $(Df)$, then $b$ is a Nash-equilibrium.
Proposition 7.6. Let $f \in F$ and let $\sigma_a : \mathbb{R}_+ \rightarrow \Delta$ be a solution of $(Df)$ such that $\lim_{t \to \infty} \sigma_a(t) = b$. Take $T > 0$. We know that for all $t \geq 0$,

$$\|\sigma_a(T + t) - \sigma_b(T)\| \leq e^{L \cdot T} \|\sigma_a(t) - b\|.$$ 

Because $\sigma_a(t) \rightarrow b$ it follows that $\sigma_a(t) \rightarrow \sigma_b(T)$. Thus $\sigma_b(T) = b$. Hence, $b$ is a rest-point of $(Df)$ and therefore, by Proposition 7.4, it is a Nash-equilibrium.

**Remark.** If $b \in \Delta$ is a rest-point, it can be verified that $\Delta \setminus \{b\}$ is $(Df)$-invariant. So, if $b$ is reachable from $a \in \Delta$ with $a \neq b$, then there exists no number $T > 0$ such that $\sigma_a(T) = b$. In words, although $b$ is reachable, one needs ‘infinite time’ to reach it from another strategy profile $a \neq b$.

So, for every $f \in F$, if $\sigma_a(t)$ converges for some $a \in \Delta$, then the limit is a Nash-equilibrium. But what if $\sigma_a(t)$ does not converge for some $f \in F$ and $a \in \Delta$? In this case, we look at the limit set of the strategy profile $a \in \Delta$.

**Definition.** Let $f \in F$. For every strategy profile $a \in \Delta$ we define the limit set of $a \in \Delta$ as,

$$L_a(f) := \{b \in \Delta \mid \lim_{m \to \infty} \sigma_a(t_m) = b \text{ with } t_m \uparrow \infty\}.$$ 

Loosely speaking, the limit set $L_a(f)$ of $a \in \Delta$ contains all strategy profiles that are approximated arbitrarily close and arbitrarily often by the orbit $O_a(f)$. It turns out that a limit set is non-empty, closed, connected and $(Df)$-invariant for every $f \in F$.

**Proposition 7.6.** Let $f \in F$. For every strategy profile $a \in \Delta$ the limit set $L_a(f)$ is

(i) Non-empty, (ii) Closed, (iii) Connected, (iv) $(Df)$-invariant.

**Proof.** (i) Let $f \in F$. Suppose $L_a(f) = \emptyset$ for some $a \in \Delta$. Then for each $b \in \Delta$ there is a number $\varepsilon(b) > 0$ and a number $T(b) \geq 0$ such that

$$t \geq T(b) \Rightarrow \sigma_a(t) \notin B_{\varepsilon(b)}(b).$$ 

Recall that $B_{\varepsilon(c)} := \{\sigma \in \Delta \mid |\sigma - c| < \varepsilon\}$. The collection $\{B_{\varepsilon(b)}(b)\}_{b \in \Delta}$ covers $\Delta$ and therefore, due to compactness of $\Delta$, there exists $b_1, \ldots, b_p \in \Delta$ such that

$$\bigcup_{\ell=1}^{p} B_{\varepsilon(b_\ell)}(b_\ell) \supseteq \Delta.$$ 

Define $T := \max\{T(b_\ell) \mid \ell = 1, \ldots, p\}$. Then $\sigma_a(t) \notin \bigcup_{\ell=1}^{p} B_{\varepsilon(b_\ell)}(b_\ell)$ for all $t > T$. This contradicts the $(Df)$-invariance of $\Delta$.

(ii) Let $f \in F$ and take $a \in \Delta$. Let $b_1, b_2, \ldots$ be a sequence in $L_a(f)$ with $b_m \rightarrow b$. Suppose $b \notin L_a(f)$. So, there is no sequence $t_m \uparrow \infty$ with $\sigma_a(t_m) \rightarrow b$. Hence, there is a number $\varepsilon > 0$ and a number $T \geq 0$ such that

$$\sigma_a(t) \notin B_{\varepsilon}(b) \text{ for all } t \geq T.$$
7.5 The limit set and stability

There exists a number \( \ell \in \mathbb{N} \) with \( b_\ell \in B_\varepsilon(b) \). Choose \( \varepsilon' < \varepsilon \) such that \( B_{\varepsilon'}(b_\ell) \subseteq B_{\varepsilon}(b) \).

Because \( b_\ell \in L_a(f) \) there exists a number \( T' > T \) such that

\[
\sigma(T') \in B_{\varepsilon'}(b_\ell) \subseteq B_{\varepsilon}(b).
\]

Contradiction. Hence, \( b \in L_a(f) \) and therefore \( L_a(f) \) is closed.

(iii) Suppose there exists an \( f \in \mathcal{F} \) such that \( L_a(f) \) is not connected for some \( a \in \Delta \). Then \( L_a(f) = A \cup A' \), with \( A \) and \( A' \) non-empty, closed and disjoint sets. Take \( b \in A \) and \( b' \in A' \). Then there are sequences \( t_m \uparrow \infty \) and \( t'_m \uparrow \infty \) with \( t'_m > t_m \) for all \( m \in \mathbb{N} \) such that

\[
\lim_{m \to \infty} \sigma_a(t_m) = b \quad \text{and} \quad \lim_{m \to \infty} \sigma_a(t'_m) = b'.
\]

Let \( B \) and \( B' \) be disjoint neighborhoods of \( A \) and \( A' \), respectively (note that a neighborhood \( B \) of a set \( A \) is an open set properly containing \( A \)). We may assume that \( \sigma_a(t_m) \in B \) and \( \sigma_a(t'_m) \in B' \) for all \( m \in \mathbb{N} \). Define for all \( m \in \mathbb{N} \) the connected set

\[
\sigma_a[t_m, t'_m] := \{ \sigma_a(t) \ | \ t \in [t_m, t'_m] \}.
\]

Then for all \( m \in \mathbb{N} \) there exist a number \( t_m < T_m < t'_m \) such that \( \sigma_a(T_m) \in \Delta \setminus (B \cup B') \). Because \( \Delta \setminus (B \cup B') \) is compact we may assume that \( \sigma_a(T_m) \to b'' \). And because \( T_m \uparrow \infty \) it follows that \( b'' \in L_a(f) \). However, \( b'' \notin A \cup A' \). Contradiction.

(iv) Let \( f \in \mathcal{F} \) and take \( a \in \Delta \). Let \( b \in L_a(f) \) and \( T > 0 \). There is a sequence \( t_m \uparrow \infty \) such that \( \lim_{m \to \infty} \sigma_a(t_m) = b \). Then \( T + t_m \uparrow \infty \) and

\[
|\sigma_a(T + t_m) - \sigma_b(T)| \leq e^{LT} ||\sigma_a(t_m) - b||.
\]

Hence, \( \lim_{m \to \infty} \sigma_a(T + t_m) = \sigma_b(T) \). And thus \( \sigma_b(T) \in L_a(f) \). \( \square \)

This chapter investigates when the limit set is a subset of the set of Nash-equilibria. In general this is not the case. This is shown by Hart and Mas-Colell (2003). They provide a class of three-person normal-form games in which each game has a unique Nash-equilibrium. They prove that every uncoupled dynamic selection process does not converge to this unique Nash-equilibrium. A dynamic selection process is said to be uncoupled if for each player \( i \in N \) the changes in the weights on each of his pure strategies \( s^k_i \) do not depend on the payoff functions \( U_j \) for all \( j \neq i \). This means that \( \hat{s}^k_i \) does depend on \( \sigma \) and on changes in the payoff function \( U_i \) only, for all \( 1 \leq k \leq K_i \) and \( i \in N \). So, a dynamic selection process in which players act naively and ignore the payoffs of the others, cannot converge to Nash-equilibria in a general normal-form game. Given an \( f \in \mathcal{F} \), the dynamic selection process defined by \( (D_f) \) is an example of such an uncoupled process. Therefore, the result of Hart and Mas-Colell tells us that for these three-person games, the limit sets do not contain a Nash-equilibrium. To be complete, we repeat the example of Hart and Mas-Colell.

**Example (Hart and Mas-Colell (2003)).** Let \( (\alpha, \beta, \gamma) \in \mathbb{R}^3_+ \). We define \( \Gamma(\alpha, \beta, \gamma) \) as the class of three-person normal-form games with two pure strategies for each player and the following payoff:
As usual, player 1 chooses the row, player 2 the column and player 3 the matrix. The normal-form game $\Gamma(1, 1, 1)$ has been introduced by Jordan (1993). Some calculations yield that every normal-form game $(\gamma; \beta; \alpha)$ has exactly one Nash-equilibrium, namely,$$
abla(\alpha, \beta, \gamma) := (\frac{\alpha}{1+\beta} \cdot s_1^1 + \frac{\beta}{1+\alpha} \cdot s_1^2 + \frac{\gamma}{1+\alpha} \cdot s_1^3, \frac{\alpha}{1+\beta} \cdot s_2^1 + \frac{\beta}{1+\alpha} \cdot s_2^2 + \frac{\gamma}{1+\alpha} \cdot s_2^3, \frac{\alpha}{1+\beta} \cdot s_3^1 + \frac{\beta}{1+\alpha} \cdot s_3^2 + \frac{\gamma}{1+\alpha} \cdot s_3^3)).$$

Take $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ and let $f \in F$. Consider the map $\mathcal{H} : [0, 1]^3 \rightarrow [0, 1]^3$ defined by

\[
(x, y, z) \mapsto (\mathcal{H}_1(x, y, z), \mathcal{H}_2(x, y, z), \mathcal{H}_3(x, y, z))
\]

where for all $x, y, z \in [0, 1]$:

\[
\begin{align*}
\mathcal{H}_1(x, y, z) &:= (1-x) \cdot f_1^1(((1-x) \cdot [\alpha - (\alpha + 1) \cdot y]_{+}) - x \cdot f_1^2((x \cdot (\alpha + 1) \cdot y - \alpha)_{+}) \\
\mathcal{H}_2(x, y, z) &:= (1-y) \cdot f_2^1(((1-y) \cdot [\beta - (\beta + 1) \cdot z]_{+}) - y \cdot f_2^2((y \cdot (\beta + 1) \cdot z - \beta)_{+}) \\
\mathcal{H}_3(x, y, z) &:= (1-z) \cdot f_3^1(((1-z) \cdot [\gamma - (\gamma + 1) \cdot x]_{+}) - z \cdot f_3^2((z \cdot (\gamma + 1) \cdot x - \gamma)_{+}).
\end{align*}
\]

Observe that $\mathcal{H}_1(x, \frac{\alpha}{1+\alpha}, z) = 0$ for every $(x, z) \in [0, 1] \times [0, 1]$. Therefore,

\[
\frac{d}{dx} \mathcal{H}_1(x, \frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta})_{|x=\frac{\gamma}{1+\gamma}} = \frac{d}{dx} \mathcal{H}_1((\frac{\gamma}{1+\gamma}, \frac{\alpha}{1+\alpha}, z)_{|z=\frac{\beta}{1+\beta}} = 0.
\]

Similarly, it follows that

\[
\frac{d}{dy} \mathcal{H}_2(x, \frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta})_{|y=\frac{\gamma}{1+\gamma}} = \frac{d}{dy} \mathcal{H}_2((\frac{\gamma}{1+\gamma}, y, \frac{\alpha}{1+\alpha})_{|y=\frac{\alpha}{1+\alpha}} = 0
\]

and that

\[
\frac{d}{dz} \mathcal{H}_3(\frac{\gamma}{1+\gamma}, y, \frac{\beta}{1+\beta})_{|y=\frac{\alpha}{1+\alpha}} = \frac{d}{dz} \mathcal{H}_3((\frac{\gamma}{1+\gamma}, \frac{\alpha}{1+\alpha}, z)_{|z=\frac{\beta}{1+\beta}} = 0.
\]

Hence, the $3 \times 3$ Jacobian matrix $J$ in $(\frac{\gamma}{1+\gamma}, \frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta})$ looks like,

\[
J = \begin{pmatrix}
0 & c & 0 \\
0 & d & 0 \\
c & 0 & e
\end{pmatrix}
\]

for some numbers $c, d, e \in \mathbb{R}$. Observe that $\det(J) = cde$. After computing the characteristic polynomial of $J$, it can be verified that $J$ has at least one eigenvalue with positive real part. According to the Poincaré-Lyapunov Theorem, this implies that the derivatives ‘lead away’ from $(\frac{\gamma}{1+\gamma}, \frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta})$ and therefore it is unstable (see e.g., Hirsch and Smale (1974) for all the details).
Combining this observation with the fact that the set of differential equations \((Df)\) equals,
\[
\dot{\sigma}_1^1 = \mathcal{H}_1(\sigma_1^1, \sigma_2^1, \sigma_3^1), \quad \dot{\sigma}_2^1 = \mathcal{H}_2(\sigma_1^1, \sigma_2^1, \sigma_3^1) \quad \text{and} \quad \dot{\sigma}_3^1 = \mathcal{H}_3(\sigma_1^1, \sigma_2^1, \sigma_3^1),
\]
yields for the Nash-equilibrium \(\eta(\alpha, \beta, \gamma)\) to be unreachable. Differently, not every limit set contains the unique Nash-equilibrium.

So, in general a limit set does not contain a Nash-equilibrium. However, our main goal is to find classes of normal-form games in which for every \(f \in \mathcal{F}\) every limit set is a subset of the set of Nash-equilibria. To achieve this goal, we search for normal-form games which have a so-called Lyapunov function for \((Df)\) (see e.g., Hirsch and Smale (1974)). Let us first repeat the definition of a Lyapunov function.

**Definition.** Let \(f \in \mathcal{F}\). A differentiable function \(L : \Delta \rightarrow \mathbb{R}\) is a Lyapunov function for \((Df)\) if it has the following two properties:

(i) \(\frac{d}{dt} L(\sigma_a(t)) \leq 0\) for every \(a \in \Delta\),

(ii) \(\frac{d}{dt} L(\sigma_a(t))|_{t=0} = 0 \Leftrightarrow a\) is a rest-point of \((Df)\).

A Lyapunov function is decreasing on the orbit \(O_a(f)\) for every \(a \in \Delta\). This implies that a Lyapunov function has a constant value on a limit set.

**Lemma 7.7.** Let \(f \in \mathcal{F}\). If \(L : \Delta \rightarrow \mathbb{R}\) is a Lyapunov function for \((Df)\) then \(L\) is constant on \(L_a(f)\) for every \(a \in \Delta\).

**Proof.** Let \(f \in \mathcal{F}\) and let \(L : \Delta \rightarrow \mathbb{R}\) be a Lyapunov function for \((Df)\). Take \(a \in \Delta\). Suppose \(L(b) < L(b')\) for certain \(b, b' \in L_a(f)\) with \(b \neq b'\).

Define \(\varepsilon := L(b') - L(b)\). There exists a number \(T \geq 0\) such that
\[
L(\sigma_a(T)) < L(b) + \frac{\varepsilon}{2}.
\]
And thus for all \(t > T\) we have that
\[
L(\sigma_a(t)) \leq L(\sigma_a(T)) < L(b) + \frac{\varepsilon}{2}.
\]
Hence, for all \(t > T\),
\[
L(\sigma_a(t)) < L(b') - \frac{\varepsilon}{2}.
\]
But \(b' \in L_a(f)\). Contradiction.

Now we can prove that for normal-form games which have a Lyapunov function for \((Df)\), every limit set is a subset of the set of Nash-equilibria.

**Proposition 7.8.** Let \(f \in \mathcal{F}\). If \(\Gamma\) has a Lyapunov function for \((Df)\), then \(L_a(f) \subseteq NE(\Gamma)\) for every \(a \in \Delta\).

**Proof.** Let \(f \in \mathcal{F}\) and take \(b \in L_a(f)\) for some \(a \in \Delta\). Suppose \(b \notin NE(\Gamma)\), then \(b\) is not a rest-point (Proposition 7.4) and therefore,
\[
\frac{d}{dt} L(\sigma_a(t))|_{t=0} < 0.
\]
But \(L\) is constant on \(L_a(f)\) (Lemma 7.7) and \(L_a(f)\) is \((Df)\)-invariant (Proposition 7.6(iv)). Contradiction.
Hence, if a normal-form game has a Lyapunov function, then the dynamics defined by \((Df)\) converge for every \(f \in F\) only towards Nash-equilibria. Therefore, the main goal of the remaining part of this section is to find structured classes of normal-form games for which there exists a Lyapunov function. But before we do so, we end this part of the section by repeating the definitions of two well-known stability concepts for dynamic selection processes, namely, Lyapunov and asymptotic stability. Let us start by giving the definition of Lyapunov stability.

**Definition.** Let \(f \in F\). A closed \((Df)\)-invariant set \(A \subseteq \Delta\) is **Lyapunov stable** under \((Df)\), if for every neighborhood \(B\) of \(A\) there is a neighborhood \(B'\) of \(A\) such that

\[
\text{if } a \in B' \cap \Delta, \text{ then } \sigma_a(t) \in B \text{ for all } t \geq 0.
\]

Lyapunov stability states that the orbit \(O_a(f)\) remains close to \(A\) whenever \(a\) is sufficiently close to \(A\). A more demanding definition of stability is asymptotic stability. Besides Lyapunov stability it also requires the notion of local attractor. This means that \(\sigma_b(t)\) converges to \(A\) for every \(b\) sufficiently close to \(A\). Let us give a more formal definition of this stability concept.

**Definition.** Let \(f \in F\). A closed \((Df)\)-invariant set \(A \subseteq \Delta\) is **asymptotically stable** under \((Df)\), if it is Lyapunov stable and there exists a neighborhood \(B\) of \(A\) such that

\[
\text{if } a \in B \setminus \Delta, \text{ then } L_a(f) \subseteq A \text{ for every } b \in B \cap \Delta.
\]

**Remark.** If \(\sigma \in \Delta\) is a strategy profile such that \(\{\sigma\}\) is Lyapunov or asymptotically stable under \((Df)\), then \(\{\sigma\}\) is in particular \((Df)\)-invariant. Hence, by Proposition 7.4, \(\sigma\) is a Nash-equilibrium.

For a connected component of the set of Nash-equilibria (Kohlberg and Mertens (1986)) the existence of a Lyapunov function for \((Df)\) closes the gap between Lyapunov and asymptotic stability. Furthermore, a connected component of \(\text{NE}(\Gamma)\) which is asymptotically stable, is minimal asymptotically stable (i.e., it does not properly contain an asymptotically stable subset). Lemma 7.9 provides a more formal statement on these two points.

**Lemma 7.9.** Let \(f \in F\) and let \(K\) be a connected component of Nash-equilibria of \(\Gamma\) which is Lyapunov stable under \((Df)\). If \(\Gamma\) has a Lyapunov function for \((Df)\), then \(K\) is minimal asymptotically stable under \((Df)\).

**Proof.** Let \(f \in F\) and let \(K\) be a connected component of \(\text{NE}(\Gamma)\) which is Lyapunov stable under \((Df)\). We first prove that \(K\) is asymptotically stable. There exists a neighborhood \(B\) of \(K\) such that

\[
\text{NE}(\Gamma) \cap \text{cl}(B) = K.
\]

We write cl\((B)\) for the closure of \(B\). Then, due to Lyapunov stability of \(K\), there exists a neighborhood \(B'\) of \(K\) such that

\[
\text{if } a \in B' \cap \Delta, \text{ then } \sigma_a(t) \in B \text{ for all } t \geq 0.
\]

So in particular, \(L_a(f) \subseteq \text{cl}(B)\) for all \(a \in B' \cap \Delta\). Because \(\Gamma\) has a Lyapunov function for \((Df)\) it follows, by Proposition 7.8, that \(L_a(f) \subseteq \text{NE}(\Gamma)\) for every \(a \in B' \cap \Delta\). Hence, \(L_a(f) \subseteq K\) for every \(a \in B' \cap \Delta\). Hence, \(K\) is a local attractor and thus asymptotically stable.
Now we prove that $K$ is minimal asymptotically stable. Suppose $A$ is a proper subset of $K$ which is asymptotically stable. Because $K$ is a component, every neighborhood $B$ of $A$ contains strategy profiles in $K \setminus A$. So, in particular $\sigma \in NE(\Gamma)$. By Proposition 7.4 this means that $\sigma$ is a rest-point of $(Df)$. Hence, $A$ is not a local attractor. Contradiction.

Before closing this part of the section we prove a result which we use throughout the forthcoming subsections. For this purpose, we first need the following definition.

**Definition.** Let $f \in F$ and let $L : \Delta \rightarrow \mathbb{R}$ be a Lyapunov function for $(Df)$. Then a set $A \subseteq \Delta$ is a local minimizer of $L$ if:

(i) $A$ is connected,
(ii) $L$ is constant on $A$,
(iii) There exists a neighborhood $B$ of $A$ such that $L(\sigma) < L(\hat{\sigma})$ for every $\sigma \in A$ and every $\hat{\sigma} \in \Delta \cap B \setminus A$.

**Proposition 7.10.** Let $f \in F$ and let $L : \Delta \rightarrow \mathbb{R}$ be a Lyapunov function for $(Df)$. Then a connected component of Nash-equilibria on which $L$ has a constant value is minimal asymptotically stable under $(Df)$ if and only if it is a local minimizer of $L$.

**Proof.**

$(\Rightarrow)$ Let $L : \Delta \rightarrow \mathbb{R}$ be a Lyapunov function for $(Df)$ and let $K$ be a connected component of Nash-equilibria which is minimal asymptotically stable under $(Df)$ and on which $L$ has a constant value, say it has value $L(K)$. Suppose $K$ is not a local minimizer of $L$. Then for every neighborhood $B$ of $A$ there exists a strategy profile $\sigma \in B \setminus K$ such that $L(\sigma) \leq L(K)$. Because $L$ is a Lyapunov function for $(Df)$ for every $f \in F$ and thus strictly decreasing on $\Delta \setminus NE(\Gamma)$, this implies that $K$ cannot be a local attractor. Contradiction.

$(\Leftarrow)$ Let $L : \Delta \rightarrow \mathbb{R}$ be a Lyapunov function for $(Df)$ and let $K$ be a connected component of Nash-equilibria which is a local minimizer of $L$, say again it has the constant value $L(K)$ on $K$. Let $B$ be a neighborhood of $K$. There exists a neighborhood $B'$ of $K$ such that $L(K) < L(\sigma)$ for every strategy profile $\sigma \in B' \setminus K$. Define

$$L_0 := \min \{ L(\sigma) \mid \sigma \in \partial(B \cap B') \},$$
$$B'' := \{ \sigma \in B \cap B' \mid L(\sigma) < L_0 + \frac{L(K)}{2} \}.$$ 

We write $\partial(A)$ for the boundary of $A$. Observe that $B''$ is a neighborhood of $K$ and because $L$ is decreasing on $B \cap B'$, it follows that

if $a \in B''$ then $\sigma_a(t) \in B$ for all $t \geq 0$.

Hence, $K$ is Lyapunov stable under $(Df)$ for every $f \in F$. By Lemma 7.9 this yields that $K$ is minimal asymptotically stable under $(Df)$ for every $f \in F$.

### 7.5.1 The limit set and stability in zero-sum games

The current subsection is devoted to investigate the limit set and stability of the dynamics $(Df)$ on the class of two-person zero-sum games. This class of normal-form games has been introduced in Von Neumann (1928). The defining feature is the symmetry in the payoffs,
We start by elaborating on the first part of equation (1). First note that

\[ a \supseteq (i) \quad \text{and} \quad c \supseteq (i) \]

\[ \text{Proof.} \]

Theorem 7.11. Let \( f \in \mathcal{F} \) and let \( \Gamma \) be a zero-sum game. Then \( L_a(f) \subseteq \text{NE}(\Gamma) \) for every \( a \in \Delta \).

\[ \text{Remark.} \]

A zero-sum game \( \Gamma \) has a so-called min-max value \( v(\Gamma) \) (Vilkas (1963)). This value is the payoff player 1 can guarantee himself and \(-v(\Gamma)\) is the payoff player 2 can guarantee himself. A strategy profile \((\sigma_1, \sigma_2) \in \Delta\) is a Nash-equilibrium of a zero-sum game \( \Gamma \) if and only if \( \sigma_1 \) and \( \sigma_2 \) are optimal strategies for player 1 and player 2, respectively (i.e., \( \sigma_1 \) guarantees the payoff \( v(\Gamma) \) and \( \sigma_2 \) guarantees the payoff \(-v(\Gamma)\)).

Every limit set is a subset of the set of Nash-equilibria on the class of zero-sum games. We prove this statement by showing that for any \( f \in \mathcal{F} \) a zero-sum game has a Lyapunov function for \((Df)\).

\[ \text{Theorem 7.11.} \]

Let \( f \in \mathcal{F} \) and let \( \Gamma \) be a zero-sum game. Then \( L_a(f) \subseteq \text{NE}(\Gamma) \) for every \( a \in \Delta \).

\[ \text{Proof.} \]

Let \( f \in \mathcal{F} \) and let \( F^k_i \) be a primitive function of \( f^k_i \) (i.e., \( F^k_i = f^k_i \)) for all \( 1 \leq k \leq K_i \) and \( i \in \{1, 2\} \). We prove that

\[ \sigma \mapsto \sum_{k=1}^{K_1} F^k_1 \circ R^k_1(\sigma) + \sum_{\ell=1}^{K_2} F^\ell_2 \circ R^\ell_2(\sigma) \]

is a Lyapunov function for \((Df)\).

Let \( \sigma : \mathbb{R}_+ \rightarrow \Delta \) be a solution of \((Df)\) and take \( 1 \leq k \leq K_1 \), then (time indices are suppressed),

\[ (F^k_1 \circ R^k_1)(\sigma) = f^k_1 \circ R^k_1(\sigma) \cdot [U_1(s^k_1, \sigma_2) - U_1(\sigma_1, \sigma_2)] \]

\[ = f^k_1 \circ R^k_1(\sigma) \cdot \left[ \sum_{\ell=1}^{K_2} \sigma^\ell_2 \cdot [U_1(s^k_1, s^\ell_2) - U_1(\sigma_1, s^\ell_2)] - \sum_{p=1}^{K_2} \sigma^p_1 \cdot [U_1(s^p_1, \sigma_2)] \right]. \quad (1) \]

We start by elaborating on the first part of equation (1). First note that

\[ \sum_{\ell=1}^{K_2} \sigma^\ell_2 \cdot [U_1(s^k_1, s^\ell_2) - U_1(\sigma_1, s^\ell_2)] = \sum_{\ell=1}^{K_2} f^\ell_2 \circ R^\ell_2(\sigma) \cdot [U_1(s^k_1, s^\ell_2) - U_1(\sigma_1, s^\ell_2)] \]

\[ - f_2 \circ R_2(\sigma) \cdot \sum_{\ell=1}^{K_2} \sigma^\ell_2 \cdot [U_1(s^k_1, s^\ell_2) - U_1(\sigma_1, s^\ell_2)] \]

\[ = \sum_{\ell=1}^{K_2} f^\ell_2 \circ R^\ell_2(\sigma) \cdot [U_1(s^k_1, s^\ell_2) - U_1(\sigma_1, s^\ell_2)] \]

\[ - f_2 \circ R_2(\sigma) \cdot [U_1(s^k_1, \sigma_2) - U_1(\sigma_1, \sigma_2)]. \]
Furthermore, note that
\[ f_2^\ell \circ R_2^\ell (\sigma) \cdot [U_1(s_1^k, s_2^\ell) - U_1(\sigma_1, s_2^\ell)] \]
\[ = \sum_{\ell=1}^{K_2} f_2^\ell \circ R_2^\ell (\sigma) \cdot [U_1(s_1^k, s_2^\ell) - U_1(\sigma_1, \sigma_2) + U_1(\sigma_1, \sigma_2) - U_1(\sigma_1, s_2^\ell)] \]
\[ = \sum_{\ell=1}^{K_2} f_2^\ell \circ R_2^\ell (\sigma) \cdot [U_1(s_1^k, s_2^\ell) - U_1(\sigma_1, \sigma_2)] + \sum_{\ell=1}^{K_2} f_2^\ell \circ R_2^\ell (\sigma) \cdot R_2^\ell (\sigma). \]

The last equality follows from the fact that \( U_1 = -U_2 \) and from the fact that,
\[ f_2^\ell \circ R_2^\ell (\sigma) \cdot [U_2(\sigma_1, s_2^\ell) - U_2(\sigma_1, \sigma_2)] = f_2^\ell \circ R_2^\ell (\sigma) \cdot R_2^\ell (\sigma) \text{ for all } 1 \leq \ell \leq K_2. \]

Hence, the first part of equation (1) equals,
\[ f_1^k \circ R_1^k (\sigma) \cdot \sum_{\ell=1}^{K_2} f_2^\ell \circ R_2^\ell (\sigma) \cdot [U_1(s_1^k, s_2^\ell) - U_1(\sigma_1, s_2^\ell)] \]
\[ = f_1^k \circ R_1^k (\sigma) \cdot \left( \sum_{\ell=1}^{K_2} f_2^\ell \circ R_2^\ell (\sigma) \cdot [U_1(s_1^k, s_2^\ell) - U_1(\sigma_1, \sigma_2)] + \sum_{\ell=1}^{K_2} f_2^\ell \circ R_2^\ell (\sigma) \cdot R_2^\ell (\sigma) \right) \]
\[ - f_1^k \circ R_1^k (\sigma) \cdot [U_1(s_1^k, \sigma_2) - U_1(\sigma_1, \sigma_2)] \cdot (f_2 \circ R_2(\sigma)) \]
\[ = f_1^k \circ R_1^k (\sigma) \cdot \sum_{\ell=1}^{K_2} f_2^\ell \circ R_2^\ell (\sigma) \cdot [U_1(s_1^k, s_2^\ell) - U_1(\sigma_1, \sigma_2)] + \sum_{\ell=1}^{K_2} f_2^\ell \circ R_2^\ell (\sigma) \cdot R_2^\ell (\sigma) \]
\[ + f_1^k \circ R_1^k (\sigma) \cdot \sum_{\ell=1}^{K_2} f_2^\ell \circ R_2^\ell (\sigma) - (f_1^k \circ R_1^k (\sigma) \cdot R_2^k (\sigma)) \cdot (f_2 \circ R_2(\sigma)). \] (2)

The last equality follows from the fact that \( f_1^k \circ R_1^k (\sigma) \cdot [U_1(s_1^k, \sigma_2) - U_1(\sigma_1, \sigma_2)] = f_1^k \circ R_1^k (\sigma) \cdot R_1^k (\sigma) \text{ for all } 1 \leq k \leq K_1. \)

Next, we elaborate on the second part of equation (1). Note that
\[ \sum_{p=1}^{K_1} \sigma^p \cdot U_1(s_1^p, \sigma_2) = \sum_{p=1}^{K_1} f_1^p \circ R_1^p (\sigma) \cdot U_1(s_1^p, \sigma_2) = f_1 \circ R_1(\sigma) \cdot \sum_{p=1}^{K_1} \sigma^p \cdot U_1(s_1^p, \sigma_2) \]
Adding equations (2) and (3) and taking the sum over follows that (4a) + (5a)

\[ = \sum_{p=1}^{K_1} f^p_1 \circ R^p_1(\sigma) \cdot U_1(s^p_1, \sigma_2) - f_1 \circ R_1(\sigma) \cdot U_1(\sigma_1, \sigma_2) \]

\[ = \sum_{p=1}^{K_1} f^p_1 \circ R^p_1(\sigma) \cdot [U_1(s^p_1, \sigma_2) - U_1(\sigma_1, \sigma_2)] \]

\[ = \sum_{p=1}^{K_1} f^p_1 \circ R^p_1(\sigma) \cdot R^p_1(\sigma). \]

Hence, the second part of equation (1) becomes,

\[ -f^k_1 \circ R^k_1(\sigma) \sum_{p=1}^{K_1} \hat{\sigma}^p \cdot U_1(s^p_1, \sigma_2) = -f^k_1 \circ R^k_1(\sigma) \cdot \sum_{p=1}^{K_1} f^p_1 \circ R^p_1(\sigma) \cdot R^p_1(\sigma). \]  \hspace{1cm} (3)

Adding equations (2) and (3) and taking the sum over \( 1 \leq k \leq K_1 \) yields,

\[
\sum_{k=1}^{K_1} F^k_1 \circ R^k_1(\sigma) = \sum_{k=1}^{K_1} \sum_{\ell=1}^{K_2} (f^k_1 \circ R^k_1(\sigma)) \cdot (f^\ell_2 \circ R^\ell_2(\sigma)) \cdot [U_1(s^k_1, s^\ell_2) - U_1(\sigma_1, \sigma_2)] \label{4a}
\]

\[ + (f_1 \circ R_1(\sigma)) \cdot \sum_{\ell=1}^{K_2} f^\ell_2 \circ R^\ell_2(\sigma) \cdot R^\ell_2(\sigma) \]  \hspace{1cm} (4b)

\[ - (f_2 \circ R_2(\sigma)) \cdot \sum_{k=1}^{K_1} f^k_1 \circ R^k_1(\sigma) \cdot R^k_1(\sigma) \]  \hspace{1cm} (4c)

\[ - (f_1 \circ R_1(\sigma)) \cdot \sum_{k=1}^{K_1} f^k_1 \circ R^k_1(\sigma) \cdot R^k_1(\sigma). \]  \hspace{1cm} (4d)

Similarly, it can be derived that

\[
\sum_{\ell=1}^{K_2} F^\ell_2 \circ R^\ell_2(\sigma) = \sum_{k=1}^{K_1} \sum_{\ell=1}^{K_2} (f^k_1 \circ R^k_1(\sigma)) \cdot (f^\ell_2 \circ R^\ell_2(\sigma)) \cdot [U_2(s^k_1, s^\ell_2) - U_2(\sigma_1, \sigma_2)] \label{5a}
\]

\[ + (f_2 \circ R_2(\sigma)) \cdot \sum_{k=1}^{K_1} f^k_1 \circ R^k_1(\sigma) \cdot R^k_1(\sigma) \]  \hspace{1cm} (5b)

\[ - (f_1 \circ R_1(\sigma)) \cdot \sum_{\ell=1}^{K_2} f^\ell_2 \circ R^\ell_2(\sigma) \cdot R^\ell_2(\sigma) \]  \hspace{1cm} (5c)

\[ - (f_2 \circ R_2(\sigma)) \cdot \sum_{\ell=1}^{K_2} f^\ell_2 \circ R^\ell_2(\sigma) \cdot R^\ell_2(\sigma). \]  \hspace{1cm} (5d)

Observe that (4b) + (5c) = 0 and that (4c) + (5b) = 0. Furthermore, because \( U_1 = - U_2 \), it follows that (4a) + (5a) = 0. Therefore,
7.5 The limit set and stability

\[
\sum_{k=1}^{K_1} F_1^k \circ R_1^k(\sigma) + \sum_{\ell=1}^{K_2} F_2^\ell \circ R_2^\ell(\sigma) = - \sum_{i=1}^{2} (f_i \circ R_i(\sigma)) \cdot \sum_{k=1}^{K_1} f_1^k \circ R_1^k(\sigma) \cdot R_1^k(\sigma).
\]

Hence, for every \( f \in \mathcal{F} \) a zero-sum game has a Lyapunov function for \((Df)\). Therefore, by Proposition 7.8, we can conclude that \( L_a(f) \subseteq \text{NE}(\Gamma) \) for every \( a \in \Delta \). \( \square \)

In Brown and Von Neumann (1950) (see also Hofbauer (2000)) it is shown that for symmetric zero-sum games every limit set is a subset of the set of Nash-equilibria for the Brown-Von Neumann-Nash dynamics. Since this dynamic selection process is a special case of the dynamics defined by \((Df)\), Theorem 7.11 generalizes this classic result to the asymmetric case. Another consequence of Theorem 7.11 is the following.

**Corollary 7.12.** Let \( \Gamma \) be a zero-sum game then \( \lim_{t \to \infty} U_1(\sigma_a(t)) = v(\Gamma) \) for every \( a \in \Delta \). \( \square \)

So, on the class of zero-sum games every limit set of the dynamic selection process defined by \((Df)\) is a subset of the set of Nash-equilibria. This gives rise to the question: ‘Which Nash-equilibria of a zero-sum game are stable?’ Before we study this problem we first have a closer look at the structure of the set of Nash-equilibria of a two-person normal-form game.

**Definition.** A subset \( \mathcal{N} \subseteq \Delta \) is exchangeable if for each pair \((\sigma_1, \sigma_2)\) and \((\delta_1, \delta_2)\) in \( \mathcal{N} \), the pairs \((\sigma_1, \delta_2)\) and \((\delta_1, \sigma_2)\) are also in \( \mathcal{N} \), i.e., the set \( \mathcal{N} \) is a Cartesian product \( \mathcal{N}_1 \times \mathcal{N}_2 \) with \( \mathcal{N}_i \subseteq \Delta_i \) for \( i = 1, 2 \). A subset \( \mathcal{N} \subseteq \text{NE}(\Gamma) \) is a Nash-component if \( \mathcal{N} \) is exchangeable and convex, and no convex and exchangeable subset of \( \text{NE}(\Gamma) \) contains \( \mathcal{N} \) properly, i.e., \( \mathcal{N} \) is maximal.

It is a well-known result that the set of Nash-equilibria of a two-person normal-form game is the irreducible union of finitely many Nash-components (Jansen (1981)) and that the set of Nash-equilibria of a zero-sum game consists of exactly one Nash-component. It turns out that this unique Nash-component is minimal asymptotically stable under \((Df)\).

**Theorem 7.13.** Let \( f \in \mathcal{F} \) and let \( \Gamma \) be a zero-sum game. Then the unique Nash-component of \( \Gamma \) is minimal asymptotically stable under \((Df)\).

**Proof.** Let \( \Gamma \) be a zero-sum game and \( \mathcal{N} \) be the unique Nash-component of \( \Gamma \). Take \( f \in \mathcal{F} \). By Theorem 7.11 we know that

\[
\mathcal{L} : \sigma \mapsto \sum_{k=1}^{K_1} F_1^k \circ R_1^k(\sigma) + \sum_{\ell=1}^{K_2} F_2^\ell \circ R_2^\ell(\sigma)
\]

with \( \hat{F}_1^k = f_1^k \) and \( \hat{F}_2^\ell = f_2^\ell \) for all \( 1 \leq k \leq K_1 \) and for all \( 1 \leq \ell \leq K_2 \), respectively, is a Lyapunov function for \((Df)\).

Define \( \mathcal{L}_0 := \sum_{k=1}^{K_1} F_1^k(0) + \sum_{\ell=1}^{K_2} F_2^\ell(0) \). Note that \( \mathcal{L}(\sigma) = \mathcal{L}_0 \) if and only if \( \sigma \in \mathcal{N} \). Furthermore, since \( \mathcal{L} : \Delta \to \mathbb{R} \) is a Lyapunov function for \((Df)\) and \( \mathcal{N} \) is the unique Nash-component of \( \Gamma \), we have that \( \mathcal{L}(\sigma) > \mathcal{L}_0 \) for every \( \sigma \in \Delta \setminus \mathcal{N} \). Hence, \( \mathcal{N} \) is a local minimizer. Because \( \mathcal{N} \) is in particular a connected component of \( \text{NE}(\Gamma) \), it follows, by Proposition 7.10, that \( \mathcal{N} \) is minimal asymptotically stable under \((Df)\). \( \square \)
A direct consequence of Theorem 7.13 is that only zero-sum games which have a unique Nash-equilibrium, like for instance ‘Matching Pennies’ (see the example below), have an asymptotically stable Nash-equilibrium.

**Corollary 7.14.** Let \( f \in \mathcal{F} \) and let \( \Gamma \) be a zero-sum game. Then a Nash-equilibrium is asymptotically stable under \((Df)\) if and only if it is the unique Nash-equilibrium of \( \Gamma \). \( \square \)

We end this subsection by looking at a classic example of a zero-sum game and investigate the orbit \( O_\alpha(f) \) for a particular \( f \in \mathcal{F} \).

**Example (Matching Pennies).** Consider the (two-person) zero-sum game \( \Gamma \) with two pure strategies for each player and the following payoff:

\[
\begin{array}{ccc}
1 & -1 & -1 \\
-1 & 1 & -1 \\
\end{array}
\]

The unique Nash-equilibrium is \( b := \left( \frac{1}{2}s_1^1 + \frac{1}{2}s_2^1, \frac{1}{2}s_1^2 + \frac{1}{2}s_2^2 \right) \). Hence, by Theorem 7.11, \( L_\alpha(f) = \{b\} \) for every \( \alpha \in \Delta \) and \( f \in \mathcal{F} \). According to Corollary 7.14 the Nash-equilibrium \( b \) is asymptotically stable under \((Df)\) for every \( f \in \mathcal{F} \).

Assume that \( f_k^i(x) := x \) for all \( 1 \leq k \leq 2 \) and \( i = 1, 2 \). It is left to the reader to verify that in this case the map

\[
\rho : t \mapsto (2 \cdot \sigma_1^1(t) - 1)^2 + (2 \cdot \sigma_2^1(t) - 1)^2
\]

is strictly decreasing on \( \mathbb{R}_+ \). So, for every \( \alpha \in \Delta \) the orbit \( O_\alpha(f) \) is a ‘spiral with center \( b \).

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**7.5.2 The limit set and stability in potential games**

In this part of the chapter we restrict ourselves to the class of potential games. This class of normal-form games has been introduced in Monderer and Shapley (1996). They introduce several concepts of potential games, namely, exact, weighted and ordinal potential games. A
common feature of these concepts is the existence of a real-valued function defined on the set of pure strategy profiles $S$, that incorporates information about the strategic possibilities of all players simultaneously. Monderer and Shapley prove that for exact and weighted potential games this real-valued function can be extended to a multilinear function defined on the polyhedron of mixed strategy profiles $\Delta$. For an ordinal potential game such an extension may not be possible (see Sela (1992) for an example). This subsection investigates the mixed extension of exact and weighted potential games which we call shortly potential games. Let us give the formal definition.

**Definition.** A normal-form game $\Gamma := (N, (\Delta_i)_{i \in N}, (U_i)_{i \in N})$ is a potential game if there exists a multilinear function $P : \Delta \rightarrow \mathbb{R}_+$ and a vector $\omega \in \mathbb{R}^n_+$ of strictly positive numbers such that for all $i \in N$, for every $\sigma_{-i} \in \Delta_{-i}$ and every $\sigma_i, \tilde{\sigma}_i \in \Delta_i$,

$$U_i(\sigma_{-i}, \sigma_i) - U_i(\sigma_{-i}, \tilde{\sigma}_i) = \omega_i \cdot [P(\sigma_{-i}, \sigma_i) - P(\sigma_{-i}, \tilde{\sigma}_i)].$$

The function $P$ is a potential for $\Gamma$.

**Remark.** It is easy to verify that strategy profiles maximizing a potential are Nash-equilibria of the potential game. Due to the multilinearity of the potential, there exists at least one pure strategy profile maximizing the potential. Hence, every potential game has pure Nash-equilibria.

The limit set is a subset of the set of Nash-equilibria on the class of potential games. We prove this by showing that a potential game has a Lyapunov function for $(Df)$.

**Theorem 7.15.** Let $f \in F$ and let $\Gamma$ be a potential game. Then $L_a(f) \subseteq NE(\Gamma)$ for every $a \in \Delta$.

**Proof.** Let $f \in F$ and let $P : \Delta \rightarrow \mathbb{R}$ be a potential for $\Gamma$. We prove that

$$\sigma \mapsto -P(\sigma)$$

is a Lyapunov function for $(Df)$.

Let $\sigma : \mathbb{R}_+ \rightarrow \Delta$ be a solution of $(Df)$, then (time indices are suppressed),

$$\dot{P}(\sigma) = \sum_{i \in N} \sum_{k=1}^{K_i} P(\sigma_{-i}, s_i^k) \cdot \dot{s}_i^k$$

$$= \sum_{i \in N} \sum_{k=1}^{K_i} P(\sigma_{-i}, s_i^k) \cdot [f_i^k \circ R_i^k(\sigma) - \sigma_i^k \cdot f_i \circ R_i(\sigma)]$$

Note that

$$\sum_{i \in N} \sum_{k=1}^{K_i} P(\sigma_{-i}, s_i^k) \cdot \sigma_i^k \cdot f_i \circ R_i(\sigma) = \sum_{i \in N} f_i \circ R_i(\sigma) \cdot \sum_{k=1}^{K_i} P(\sigma_{-i}, s_i^k) \cdot \sigma_i^k$$

$$= \sum_{i \in N} f_i \circ R_i(\sigma) \cdot P(\sigma).$$
Therefore,
\[
\sum_{i \in N} \sum_{k=1}^{K_i} P(\sigma_{-i}, s_i^k) \cdot [f_i^k \circ R_i^k(\sigma) - \sigma_i^k \cdot f_i \circ R_i(\sigma)]
\]
\[
= \sum_{i \in N} \sum_{k=1}^{K_i} P(\sigma_{-i}, s_i^k) \cdot f_i^k \circ R_i^k(\sigma) - P(\sigma) \cdot \sum_{\ell=1}^{K_i} f_i^\ell \circ R_i^\ell(\sigma)
\]
\[
= \sum_{i \in N} \sum_{k=1}^{K_i} f_i^k \circ R_i^k(\sigma) \cdot [P(\sigma_{-i}, s_i^k) - P(\sigma)].
\]

Because \( f_i^k \circ R_i^k(\sigma) > 0 \iff R_i^k(\sigma) > 0 \iff P(\sigma_{-i}, s_i^k) > P(\sigma) \) for all \( 1 \leq k \leq K_i \) and \( i \in N \) it follows that
\[
-\dot{P}(\sigma) = -\sum_{i \in N} \sum_{k=1}^{K_i} f_i^k \circ R_i^k(\sigma) : [P(\sigma_{-i}, s_i^k) - P(\sigma)]_+.
\]

Hence, for every \( f \in F \) a potential game has a Lyapunov function for \((Df)\) and therefore, by Proposition 7.8, \( L_a(f) \subseteq NE(\Gamma) \) for every \( a \in \Delta \).

In Subsection 7.5.1 we prove that on the class of (two-person) zero-sum games the map
\[
\sigma \mapsto \sum_{k=1}^{K_1} F_1^k \circ R_1^k(\sigma) + \sum_{\ell=1}^{K_2} F_2^\ell \circ R_2^\ell(\sigma)
\]
in which \( F_1^k = f_1^k \) and \( F_2^\ell = f_2^\ell \) for all \( 1 \leq k \leq K_1 \) and \( 1 \leq \ell \leq K_2 \), respectively, is a Lyapunov function for \( (Df) \). However, this is not true on the class of (two-person) potential games. This statement is a straightforward consequence of the following theorem.

**Theorem 7.16.** Let \( f \in F \). If \( \Gamma \) is a normal-form game for which the map
\[
\mathcal{L} : \sigma \mapsto \sum_{i \in N} \sum_{k=1}^{K_i} F_i^k \circ R_i^k(\sigma)
\]
in which \( F_i^k = f_i^k \) for all \( 1 \leq k \leq K_i \) and \( i \in N \), is a Lyapunov function for \((Df)\), then \( \Gamma \) has a single connected component of Nash-equilibria.

**Proof.** Take \( f \in F \) and let \( \Gamma \) be a normal-from game for which \( \mathcal{L} : \Delta \longrightarrow \mathbb{R} \) is a Lyapunov function for \((Df)\). Define
\[
\mathcal{L}_0 := \sum_{i \in N} \sum_{k=1}^{K_i} F_i^k(0).
\]
Observe that \( \mathcal{L}(\sigma) = \mathcal{L}_0 \) if and only if \( \sigma \) is a Nash-equilibrium and therefore \( NE(\Gamma) = \{ \sigma \in \Delta \mid \mathcal{L}(\sigma) = \mathcal{L}_0 \} \). Furthermore, since \( \mathcal{L} : \Delta \longrightarrow \mathbb{R} \) is a Lyapunov function for \((Df)\) we have that \( \mathcal{L}(\sigma) > \mathcal{L}_0 \) for every \( \sigma \in \Delta \setminus NE(\Gamma) \).
Suppose there exists a number $M \geq 2$ such that $NE(\Gamma) = \bigcup_{m=1}^{M} \mathcal{K}_m$ with $\mathcal{K}_m$ a connected component for all $1 \leq m \leq M$. Define, for all $1 \leq m \leq M$,

$$A_m := \{ a \in \Delta \mid L_a(f) \subseteq \mathcal{K}_m \}.$$ 

It can be readily seen that $A_m$ is $(Df)$-invariant and non-empty for all $1 \leq m \leq M$. Furthermore, observe that $\mathcal{K}_m$ is a local minimizer of $L$. Therefore, by Proposition 7.10, it follows that for all $1 \leq m \leq M$ the component $\mathcal{K}_m$ is minimal asymptotically stable and thus there exists a neighborhood $B_m$ of $\mathcal{K}_m$ such that $\mathcal{K}_m \subseteq B_m \subseteq A_m$. Finally, observe that since $\bigcup_m \mathcal{K}_m = NE(\Gamma)$ and $L_a(f) \subseteq NE(\Gamma)$ for every $a \in \Delta$ we have that $\bigcup_m A_m = \Delta$.

To obtain a contradiction, we first prove that $A_m$ is closed for all $1 \leq m \leq M$. Take $1 \leq m \leq M$ and let $a_1, a_2, \ldots$ be a sequence in $A_m$ such that $a_p \to a$. Suppose $a \notin A_m$, then $a \in A_{m'}$ for some $1 \leq m' \neq m \leq M$. There exists a number $T > 0$ such that $\sigma_a(T) \in B_{m'}$. Select $\varepsilon > 0$ such that $B_{\varepsilon}(\sigma_a(T)) \subseteq B_{m'}$ and take $p \in \mathbb{N}$ such that $|a_p - a| < \varepsilon$. Then

$$\|\sigma_{a_p}(T) - \sigma_a(T)\| \leq e^{L \cdot T} \|a_p - a\| < \varepsilon.$$

Hence, $\sigma_{a_p}(T) \in B_{\varepsilon}(\sigma_a(T)) \subseteq B_{m'} \subseteq A_{m'}$. But $L_{a_p}(f) \subseteq A_{m'}$. This contradicts the $(Df)$-invariance.

So, $A_m$ is a closed set for all $1 \leq m \leq M$. However, since $A_m$ is non-empty for all $1 \leq m \leq M$ and $\bigcup_m A_m = \Delta$ it follows that $A_m \cap A_{m'} \neq \emptyset$ for some $1 \leq m \neq m' \leq M$. This means that for a strategy profile $a \in A_m \cap A_{m'}$, it holds that $L_a(f) \subseteq \mathcal{K}_m \cap \mathcal{K}_{m'}$. But $\mathcal{K}_m \cap \mathcal{K}_{m'} = \emptyset$. Contradiction.

Given the fact that in general the set of Nash-equilibria of a potential game is not connected, Theorem 7.16 yields that for every $f \in \mathcal{F}$ the map stated in (6) in which $\hat{F}^k_i = f^k_i$ for all $1 \leq k \leq K_i$ and $i \in \mathbb{N}$ is not a Lyapunov function for the class of potential games. Hence, for this class of normal-form games we indeed need a Lyapunov function different from the one for the class of zero-sum games.

The property that for potential games the map $\sigma \mapsto -\mathcal{P}(\sigma)$ is a Lyapunov function for $(Df)$ in fact states that the dynamics defined by $(Df)$ satisfy the property which Sandholm (2001) calls *positive correlation*. Since, for every $f \in \mathcal{F}$ the Nash-equilibria are exactly the rest-points of $(Df)$ (Sandholm calls this property *non-comparability*) we can use the results by Sandholm for stability of Nash-equilibria on the class of potential games. To do so, we first repeat a result by Sandholm (2001). It states that a potential $\mathcal{P} : \Delta \rightarrow \mathbb{R}$ has a constant value on a smoothly connected component $\mathcal{K}$ of Nash-equilibria. The set $\mathcal{K}$ is smoothly connected if for all $\sigma, \sigma' \in \mathcal{K}$ there exists a continuous and piecewise differentiable curve $\alpha : [0, 1] \rightarrow \mathcal{K}$ such that $\alpha(0) = \sigma$ and $\alpha(1) = \sigma'$. For the sake of completeness we give also a proof of this statement.

**Proposition 7.17.** Let $\Gamma$ be a potential game with potential $\mathcal{P} : \Delta \rightarrow \mathbb{R}$. If $\mathcal{K}$ is a smoothly connected component of Nash-equilibria, then $\mathcal{P}$ is constant on $\mathcal{K}$.

**Proof.** Let $\mathcal{K}$ be a smoothly connected component of $NE(\Gamma)$ and take $\sigma, \sigma' \in \mathcal{K}$. There exists a continuous and piecewise differentiable curve $\alpha : [0, 1] \rightarrow \mathcal{K}$ such that $\alpha(0) = \sigma$ and $\alpha(1) = \sigma'$. Define
Suppose $\mathcal{P}(\hat{\sigma}) > \mathcal{P}(\sigma)$. Then there exists a number $\hat{\tau} \in (0, 1)$ such that $\frac{d}{d\tau} \gamma(\tau)_{[\tau = \hat{\tau}]} > 0$. Therefore,

$$0 < \frac{d}{d\tau} \gamma(\tau)_{[\tau = \hat{\tau}]} = \frac{d}{d\tau} \mathcal{P} \circ \alpha(\tau)_{[\tau = \hat{\tau}]} = \langle \nabla \mathcal{P}(\alpha(\hat{\tau})), \frac{d}{d\tau} \alpha(\tau)_{[\tau = \hat{\tau}]} \rangle,$$

in which $\nabla \mathcal{P} : \Delta \rightarrow \mathbb{R}^{\mid S\mid}$ denotes the gradient of $\mathcal{P}$ and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^{\mid S\mid}$.

Define $r := \frac{d}{d\tau} \alpha(\tau)_{[\tau = \hat{\tau}]}$. Then $r \neq 0$ (according to (6)). Furthermore, note that $r_{k} \geq 0$ whenever $\alpha(\hat{\tau})_{k} = 0$ and that $r_{k} \leq 0$ whenever $\alpha(\hat{\tau})_{k} = 1$ (otherwise the curve $\alpha$ would leave $\Delta$ and time $\hat{\tau}$). Select $h > 0$ such that $\alpha(\hat{\tau}) + h \cdot r_{i} \in \Delta$ and let $i \in N$. Then

$$\sum_{k=1}^{K_{i}} \mathcal{P}(\alpha(\hat{\tau})_{-i}, s_{i}^{k}) \cdot (\alpha(\hat{\tau})_{i}^{k} + h \cdot r_{i}^{k}) = \mathcal{P}(\alpha(\hat{\tau})) + h \cdot \sum_{k=1}^{K_{i}} \mathcal{P}(\alpha(\hat{\tau})_{-i}, s_{i}^{k}) \cdot r_{i}^{k}$$

$$= \mathcal{P}(\alpha(\hat{\tau})) + h \cdot \sum_{k=1}^{K_{i}} \nabla \mathcal{P}(\alpha(\hat{\tau}))_{i}^{k} \cdot \nabla \alpha(\hat{\tau})_{i}^{k}. \quad (7)$$

Because $\alpha(\hat{\tau}) \in K \subseteq \text{NE}(\Gamma)$ we have that $\alpha(\hat{\tau})_{i} \in \beta_{i}(\alpha(\hat{\tau})_{-i})$. By definition, this means that

$$\mathcal{P}(\alpha(\hat{\tau})_{-i}, \alpha(\hat{\tau})_{i} + h \cdot r_{i}) \leq \mathcal{P}(\alpha(\hat{\tau})_{-i}, \alpha(\hat{\tau})_{i}) = \mathcal{P}(\alpha(\hat{\tau})).$$

Combining this observation with equation (7) yields,

$$\sum_{k=1}^{K_{i}} \nabla \mathcal{P}(\alpha(\hat{\tau}))_{i}^{k} \cdot (\frac{d}{d\tau} \alpha(\tau)_{[\tau = \hat{\tau}]))_{i}^{k} \leq 0.$$

Hence, we can conclude that $\langle \nabla \mathcal{P}(\alpha(\hat{\tau})), \frac{d}{d\tau} \alpha(\tau)_{[\tau = \hat{\tau}]} \rangle \leq 0$. But this contradicts equation (6). As a result, the potential $\mathcal{P}$ is constant on $K_{i}$.

Combining the fact that $\sigma \mapsto - \mathcal{P}(\sigma)$ is a Lyapunov function for $(Df)$ with the results of Proposition 7.17 and Proposition 7.10, the following theorem is proved.

**Theorem 7.18 (Sandholm (2001)).** Let $f \in F$ and let $\Gamma$ be a potential game with potential $\mathcal{P} : \Delta \rightarrow \mathbb{R}$. A smoothly connected component of Nash-equilibria of $\Gamma$ is minimal asymptotically stable under $(Df)$ if and only if it is a local maximizer of $\mathcal{P}$ (i.e., a local minimizer of $- \mathcal{P}$).

Since, no connected set of completely mixed strategy profiles can be a local maximizer of a potential, the following result can be derived from Theorem 7.18 and Lemma 7.9.

**Corollary 7.19.** Let $f \in F$. If $\Gamma$ is a potential game, then no smoothly connected component of $\text{NE}(\Gamma)$ contained in the interior of $\Delta$ is Lyapunov stable.

Because a strict Nash-equilibrium is in particular a local maximizer of a potential, Theorem 7.18 yields also the following result.

**Corollary 7.20.** Let $f \in F$. If $\Gamma$ is a potential game, then a Nash-equilibrium of $\Gamma$ is asymptotically stable under $(Df)$ if and only if it is strict.
7.5 The limit set and stability

7.5.3 Strategic equivalence

In this subsection we extend some of the results of the previous two subsections to strategic equivalent games. We start by giving the definition of the concept of strategic equivalence for non-cooperative games. Given a set of players \( N \) and mixed strategy spaces \((\Delta_i)_{i \in N}\), we write \( \mathcal{G} \) for the set of all non-cooperative games with player set \( N \) and (not necessarily multilinear) payoff functions \((U_i)_{i \in N}\) defined on the polyhedron \( \Delta \).

**Definition.** The non-cooperative games \( \Gamma \) and \( \Gamma' \) in \( \mathcal{G} \) with payoff functions \((U_i)_{i \in N}\) and \((U'_i)_{i \in N}\) respectively, are **strategic equivalent** if there exists a strictly positive vector \( \lambda \in \mathbb{R}^+_n \) and (arbitrary) functions \( V_i : \Delta_{-i} \to \mathbb{R} \) for all \( i \in N \) such that for every \( \sigma \in \Delta \),

\[
U'_i(\sigma) = \lambda_i U_i(\sigma) + V_i(\sigma_{-i}) \quad \text{for all } i \in N.
\]

We write \( \Gamma \sim \Gamma' \) if and only if \( \Gamma \) and \( \Gamma' \) are strategic equivalent.

Indeed, \( \sim \) defines an equivalence relation on \( \mathcal{G} \). Furthermore, it is straightforward and not surprising that \( NE(\Gamma') = NE(\Gamma) \) whenever \( \Gamma \sim \Gamma' \). A similar result holds for the dynamics studied in this chapter, as we now demonstrate. Given \( \Gamma \in \mathcal{G} \) and \( f \in \mathcal{F} \), we write \( (Df)_\Gamma \) for the set of differential equations \( (Df) \) with respect to the normal-form game \( \Gamma \).

**Proposition 7.21.** Let \( \Gamma, \Gamma' \in \mathcal{G} \) such that \( \Gamma \sim \Gamma' \) and let \( f \in \mathcal{F} \), then there exists a \( g \in \mathcal{F} \) such that \( (Df)_\Gamma = (Dg)_{\Gamma'} \).

**Proof.** Let \( \Gamma, \Gamma' \in \mathcal{G} \), with payoff functions \((U_i)_{i \in N}\) and \((U'_i)_{i \in N}\) respectively, such that \( \Gamma \sim \Gamma' \). Then there exist a strictly positive vector \( \lambda \in \mathbb{R}^+_n \) and functions \( V_i : \Delta_{-i} \to \mathbb{R} \) for \( i \in N \) such that \( U'_i(\sigma) = \lambda_i U_i(\sigma) + V_i(\sigma_{-i}) \) for all \( i \in N \).

Let \( 1 \leq k \leq K_i \) and \( i \in N \). Observe that

\[
[U'_i(\sigma_{-i}, s^k_i) - U'_i(\sigma_{-i}, \sigma_i)]_+ = [\lambda_i U_i(\sigma_{-i}, s^k_i) + V_i(\sigma_{-i}) - (\lambda_i U_i(\sigma_{-i}, \sigma_i) + V_i(\sigma_{-i}))]_+ = \lambda_i [U_i(\sigma_{-i}, s^k_i) - U_i(\sigma_{-i}, \sigma_i)]_+.
\]

Hence, \( R^k_i(\sigma_\Gamma) = \lambda_i R^k_i(\sigma_\Gamma) \) for all \( 1 \leq k \leq K_i \) and \( i \in N \).

Let \( f \in \mathcal{F} \) and define \( g^k_i := f^k_i \circ \Lambda_i \) for all \( 1 \leq k \leq K_i \) and \( i \in N \), where \( \Lambda_i : x \mapsto \lambda_i x \) for all \( i \in N \). Clearly, we have \( \{g^k_i\}_{1 \leq k \leq K_i, i \in N} \in \mathcal{F} \) and furthermore,

\[
g^k_i \circ R^k_i(\sigma_\Gamma) = f^k_i \circ R^k_i(\sigma_\Gamma) \quad \text{for all } 1 \leq k \leq K_i \text{ and } i \in N.
\]

Hence, \( (Dg)_\Gamma = (Df)_{\Gamma'} \). \( \square \)

Combining Proposition 7.21 along with Theorem 7.11 and Theorem 7.15 one can directly verify the following result.

**Corollary 7.22.** Let \( \Gamma, \Gamma' \in \mathcal{G} \) such that \( \Gamma \sim \Gamma' \) and let \( f \in \mathcal{F} \).

(i) If \( \Gamma \) is a zero-sum game, then \( L_\alpha(f) \subseteq NE(\Gamma') \) for every \( \alpha \in \Delta \).

(ii) If \( \Gamma \) is a potential game, then \( L_\alpha(f) \subseteq NE(\Gamma') \) for every \( \alpha \in \Delta \). \( \square \)

Observe that the second part of Corollary 7.22 can also be derived from the fact that if \( \Gamma \sim \Gamma' \) and \( \Gamma \) is a potential game, then \( \Gamma' \) is also a potential game.
7.6 Example

Recall from Subsection 7.5.2 that on the class of potential games only strict Nash-equilibria are asymptotic stable under \((Df)\) for any \(f \in \mathcal{F}\). Since, also for the replicator dynamics only strict Nash-equilibria are asymptotic stable (Ritzberger and Weibull (1995)), it may be of interest to point out that the dynamics defined by \((Df)\) still ‘behave’ differently on the class of potential games than the replicator dynamics. To do so, we need the concept of a robust Nash-equilibrium (Okada (1983)) which is a best reply to a neighborhood of itself, but not necessarily unique. For a more precise definition we refer to Section 7.3. Ritzberger and Weibull (1995) prove that a robust Nash-equilibrium is Lyapunov stable under the replicator dynamics. However the forthcoming example of a potential game, illustrates that in general this is not true for the dynamics defined by \((Df)\).

Additionally, this example is used to illustrate three more phenomena. First of all, we use the example to emphasize the problem between asymptotic stability and Nash-equilibria. By the results of Corollary 7.14 and Corollary 7.20 it follows that many normal-from games do not have an asymptotically stable Nash-equilibrium. Of course, this is due to the fact the smallest possible asymptotic stable set is a connected component of Nash-equilibria (Lemma 7.9).

Nevertheless, although such a component may be asymptotically stable, a proper subset of Nash-equilibria may be selected by the dynamics defined by \((Df)\), but the concept of asymptotic stability cannot distinguish it from the connected component. Secondly, recall from Proposition 7.3 that the (relative) interior \(\text{int}(\Delta)\) of \(\Delta\) is \((Df)\)-invariant for every \(f \in \mathcal{F}\). Now, assume that \(b \in \Delta\) is reachable under \((Df)\) from a strategy profile \(a \in \text{int}(\Delta)\). Then we have a sequence \(\{\sigma_a(t)\}_{t \in \mathbb{R}_+}\) of completely mixed strategy profiles converging to \(b\). This might suggest for \(b \in \Delta\) to be a perfect Nash equilibrium (see Subsection 7.3 for the definition). We show in the example that this is not the case. Here, for several \(f \in \mathcal{F}\) this limit \(b\) may not be a perfect Nash-equilibrium. This means that although \(\sigma_a(t) \to b\), the strategy profile \(b\) is for every \(t \in \mathbb{R}_+\) not an element of the best reply correspondence \(\beta(\sigma_a(t))\) (i.e., for every \(t \in \mathbb{R}_+\), there exists a player \(i \in N\) such that \(b_i \notin \beta_i(\sigma_a(t)_{-i})\)). Finally, we use this example to show that for a given strategy profile \(a \in \Delta\) the limit set \(L_a(f)\), and therefore the orbit \(O_a(f)\), depends on the choice of the functions \(f \in \mathcal{F}\).

Example (Potential game). Consider the two-person potential game \(\Gamma\) with two pure strategies for player 1 and three pure strategies for player 2 and the following payoff:

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<tr>
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</tbody>
</table>

Every Nash-equilibrium of \(\Gamma\) is contained in the connected component

\[
\mathcal{K} = \{s_1^i \times \Delta_2\} \cup \{\Delta_1 \times [\sigma_2^3 = 0]\}.
\]

The set of perfect and the set of robust Nash-equilibria of \(\Gamma\) coincide and are given by (see Figure 17)

\[
\mathcal{PR} = \{s_1^i \times \Delta_2\} \cap \{\Delta_1 \times [\sigma_2^3 = 0]\} = \{s_1^i \times [\sigma_2^3 = 0]\}.
\]
7.6 Example

The notation $[\sigma_2^0 = 0]$ is an abbreviation of $\{\sigma_2 \in \Delta_2 \mid \sigma_2^0 = 0\}$. In this example, we distinguish between two different cases:

Case (1). First, take $f \in \mathcal{F}$ with the additional assumptions that

$$f_1^1(x) = x^1, \ f_2^1(x) = \alpha \cdot x^1 \quad \text{and} \quad f_2^2(x) = \beta \cdot x^2$$

for some numbers $\alpha, \beta, \kappa > 0$. Observe, that there are no extra assumptions on $f_2^2$ and on $f_2^3$.

Since $R^1_2(\sigma) = R^3_2(\sigma) = 0$ for every $\sigma \in \Delta$, the set of differential equations $(Df)$ equals:

$$\dot{\sigma}_1^1 = (1 - \sigma_1^1) \cdot [(1 - \sigma_1^1) \cdot \sigma_2^1]^{\kappa},$$
$$\dot{\sigma}_2^1 = (\alpha - (\alpha + \beta) \cdot \sigma_2^1) \cdot [(1 - \sigma_1^1) \cdot \sigma_2^1]^{\kappa},$$
$$\dot{\sigma}_2^2 = (\beta - (\alpha + \beta) \cdot \sigma_2^2) \cdot [(1 - \sigma_1^1) \cdot \sigma_2^1]^{\kappa}.$$

Note that $\dot{\sigma}_2^2 = -\dot{\sigma}_2^1$ and that $\dot{\sigma}_2^3 = -[\dot{\sigma}_2^1 + \dot{\sigma}_2^2]$. Take $a \in \Delta \setminus \mathcal{K}$ and let $\sigma : \mathbb{R}_+ \to \Delta$ be the (unique) solution of $(Df)$ with $\sigma(0) = a$. Then, by some straightforward calculations, it follows that for all $t \geq 0$,

$$\sigma_1^1(t) = 1 - \frac{1 - a_1^1}{[(1 + \alpha + \beta) \cdot (1 - a_1^1) \cdot (1 - a_2^1) \cdot (1 - a_2^2) \cdot (t + 1)]^{\alpha + \beta}},$$
$$\sigma_2^1(t) = \frac{\alpha}{\alpha + \beta} - \left(\frac{\alpha}{\alpha + \beta} - a_2^1\right) \left(\frac{1 - \sigma_1^1(t)}{1 - a_2^1}\right)^{\alpha + \beta},$$
$$\sigma_2^2(t) = \frac{\beta}{\alpha + \beta} - \left(\frac{\beta}{\alpha + \beta} - a_2^2\right) \left(\frac{1 - \sigma_1^1(t)}{1 - a_2^2}\right)^{\alpha + \beta}.$$

Hence, $\sigma_1^1(t) \to 1$ whenever $t \to \infty$. Therefore, it follows that

$$\lim_{t \to \infty} \sigma_1^1(t) = (1, 0) \quad \text{and} \quad \lim_{t \to \infty} \sigma_2^1(t) = (\frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta}, 0).$$

So, given $\alpha, \beta > 0$, only the robust Nash-equilibrium $b(\alpha, \beta) := (s_1^1, s_2^1, s_2^2, \frac{\beta}{\alpha + \beta})$ is reachable from every $a \in \Delta \setminus \mathcal{K}$, i.e., which is not a rest-point and therefore only $\{b(\alpha, \beta)\}$ is Lyapunov stable. Observe that for all $t \geq 0$,

$$(\beta - (\alpha + \beta) \cdot a_2^2) \cdot \sigma_2^1(t) - (\alpha - (\alpha + \beta) \cdot a_1^1) \cdot \sigma_2^2(t) = \beta \cdot a_2^2 - \alpha \cdot a_2^2.$$ 

This means that for every $a \in \Delta$ the orbit $O_a(f)$ is a straight line (see Figure 17).

The connected component $\mathcal{K}$ is minimal asymptotically stable, and therefore $b(\alpha, \beta)$ is not asymptotically stable for every $\alpha, \beta > 0$. So, although, given $\alpha, \beta > 0$, the limit set $L_a(f)$ is exactly the Nash-equilibrium $b(\alpha, \beta)$ for every $a \in \Delta \setminus \mathcal{K}$, this Nash-equilibrium is not selected by the concept of asymptotic stability. This illustrates that the notion of local attractor, and by that the concept of asymptotic stability, may be too demanding for evolutionary selection dynamics, since it does not select $b(\alpha, \beta)$ from the other Nash-equilibria.
Case (2). Second, take \( f \in \mathcal{F} \) with the additional assumptions that
\[
\begin{align*}
f_1^1(x) &= x^\kappa, \quad f_1^2(x) = \alpha \cdot x^{\kappa+1} \quad \text{and} \quad f_2^2(x) = \beta \cdot x^{\kappa+1}
\end{align*}
\]
for some numbers \( \alpha, \beta, \kappa > 0 \). Again there are no extra assumptions on \( f_1^2 \) and on \( f_2^1 \). Since \( R_1^2(\sigma) = R_2^2(\sigma) = 0 \) for every \( \sigma \in \Delta \), the set of differential equations (\( \mathcal{D} f \)) becomes,
\[
\begin{align*}
\dot{\sigma}_1^1 &= (1 - \sigma_1^1) \cdot [(1 - \sigma_1^1) \cdot \sigma_2^1]^{\kappa}, \\
\dot{\sigma}_2^1 &= -[(\alpha + \beta) \cdot \sigma_2^1 \cdot [(1 - \sigma_1^1) \cdot \sigma_2^1]^{\kappa+1}.
\end{align*}
\]
Note that \( \dot{\sigma}_1^2 = -\dot{\sigma}_1^1 \) and that \( [\dot{\sigma}_1^2 \dot{\sigma}_2^2] = -\dot{\sigma}_3^2 \). Take \( a \in \Delta \setminus \mathcal{K} \) and let \( \sigma : \mathbb{R}_+ \to \Delta \) be the (unique) solution of (\( \mathcal{D} f \)) with \( \sigma(0) = a \). Then, by some straightforward calculations, it follows that for all \( t \geq 0 \),
\[
\sigma_2^1(t) = \frac{a_2^2}{a_2^1 \cdot (\alpha + \beta) \cdot (\sigma_1^1(t) - a_1^1)} + 1.
\]
Furthermore, we have again that for all \( t \geq 0 \),
\[
(\beta - (\alpha + \beta) \cdot a_2^2) \cdot \sigma_2^2(t) = (\alpha - (\alpha + \beta) \cdot a_2^1) \cdot \sigma_2^1(t) = \beta \cdot a_2^1 - \alpha \cdot a_2^2
\]
which means that the orbit \( O_a(f) \) is a straight line. Combining these two results yields that \( \sigma(t) \) converges to some strategy profile \( b \in \mathcal{K} \). However, \( b_2^2 = \lim_{t \to \infty} \sigma_2^2(t) \neq 0 \) and therefore \( b \) is not a perfect Nash-equilibrium.

Observe that in Case (1) and in Case (2) the limit set \( L_a(f) \) for a given strategy profile \( a \in \Delta \) depends on the choice of \( \alpha \) and \( \beta \) and therefore on the choice of the functions \( f \in \mathcal{F} \).
References


References


184

References


References


Index

A
Agenda ........................................ 142
Agents ........................................ 45, 71
Allocation .................................... 71
n-partition ................................... 71
Asymptotic stability ........................ 166
Local attractor ................................. 166

B
Bargaining set ................................. 19, 51
Best reply ..................................... 154
Best reply correspondence ............. 154
Bi-matrix game ................................. 2

C
Capacity ......................................... 133
Individual capacity ....................... 137
Carrier ........................................ 155
Common knowledge ...................... 1
Connected components .................. 156
Consistency .................................. 117
Core ............................................. 15
Core stability ................................ 17
Exactness ..................................... 17
Extendability ................................. 17
Largeness ..................................... 17
Cost-games .................................. 138
Balancedness ................................. 138
Concavity .................................. 138
Core ........................................... 138
Subadditivity ................................. 138
Counter-objection .......................... 20, 50
Counter-value ................................. 29

D
Davis and Maschler reduced game .... 102
Direct costs .................................. 140
Dynamics ....................................... 160
Brown-Von Neumann-Nash ............ 160
Dynamic selection process ............. 159

E
Economies ..................................... 71
Efficiency ..................................... 14
Envy-freeness ................................ 73
Envy-free allocation ...................... 73
Envy-graph ................................... 84
Strong arc ................................... 84
Weak arc ..................................... 84
Envy-measure ................................. 74
Excess ......................................... 99
Exchange economies
with Indivisible goods ................. 45
with Land .................................. 143
Feasible schedules ...................... 133

G
Game Theory .................................. 1
Cooperative ................................ 4, 14
Non-cooperative ......................... 2, 154

I
Improvement .................................. 76, 143
Strong improvement .................... 46
Unilateral improvement ................. 155
Weak improvement ......................... 46
Imputation set ............................... 15
Indispensability ............................ 119
Indivisible goods ......................... 45, 71
Initial endowments ....................... 45, 143
Interior ....................................... 160
Invariance ................................. 160

J
Jobs ............................................ 133, 137
Completion-time ...................... 133, 137
<table>
<thead>
<tr>
<th>Page</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>190</td>
<td>Cost-coefficients .......... 133, 137</td>
</tr>
<tr>
<td></td>
<td>Processing demands .......... 133, 137</td>
</tr>
<tr>
<td></td>
<td><strong>K</strong></td>
</tr>
<tr>
<td></td>
<td>Kernel ........................ 20</td>
</tr>
<tr>
<td></td>
<td>Maximum surplus .............. 20</td>
</tr>
<tr>
<td></td>
<td>Positive kernel ............. 26</td>
</tr>
<tr>
<td></td>
<td>Kink functions .............. 75</td>
</tr>
<tr>
<td></td>
<td>Kink point ................... 75</td>
</tr>
<tr>
<td></td>
<td><strong>L</strong></td>
</tr>
<tr>
<td></td>
<td>Land ........................ 143</td>
</tr>
<tr>
<td></td>
<td>Limit Set .................... 162</td>
</tr>
<tr>
<td></td>
<td>Local minimizer .............. 167</td>
</tr>
<tr>
<td></td>
<td>Lyapunov function .......... 165</td>
</tr>
<tr>
<td></td>
<td>Lyapunov stability .......... 166</td>
</tr>
<tr>
<td></td>
<td><strong>M</strong></td>
</tr>
<tr>
<td></td>
<td>Matching pennies .......... 172</td>
</tr>
<tr>
<td></td>
<td>Minimal obligation .......... 119</td>
</tr>
<tr>
<td></td>
<td>Minimal rights first ........ 119</td>
</tr>
<tr>
<td></td>
<td>Money distribution .......... 71</td>
</tr>
<tr>
<td></td>
<td>Monotonicity ................ 113</td>
</tr>
<tr>
<td></td>
<td>Cost-monotonicity .......... 113, 118</td>
</tr>
<tr>
<td></td>
<td>PMAS ........................ 117</td>
</tr>
<tr>
<td></td>
<td>Population-monotonicity ..... 116</td>
</tr>
<tr>
<td></td>
<td>Revenue-monotonicity .......... 115</td>
</tr>
<tr>
<td></td>
<td><strong>N</strong></td>
</tr>
<tr>
<td></td>
<td>Nash-component .............. 171</td>
</tr>
<tr>
<td></td>
<td>Nash-equilibrium .......... 155</td>
</tr>
<tr>
<td></td>
<td>Perfect .................. 157</td>
</tr>
<tr>
<td></td>
<td>Robust .................. 158</td>
</tr>
<tr>
<td></td>
<td>Strict .................. 157</td>
</tr>
<tr>
<td></td>
<td>Neighborhood ............... 163</td>
</tr>
<tr>
<td></td>
<td>Normal-form games .......... 154</td>
</tr>
<tr>
<td></td>
<td>Nucleolus .................. 100</td>
</tr>
<tr>
<td></td>
<td>Null-player ................ 100</td>
</tr>
<tr>
<td></td>
<td><strong>O</strong></td>
</tr>
<tr>
<td></td>
<td>Objection ................... 19, 50</td>
</tr>
<tr>
<td></td>
<td>Justified objection .......... 20, 50</td>
</tr>
<tr>
<td></td>
<td>Objection-value ............ 30</td>
</tr>
<tr>
<td></td>
<td>Orbit .................... 160</td>
</tr>
<tr>
<td></td>
<td><strong>P</strong></td>
</tr>
<tr>
<td></td>
<td>Pareto efficiency .......... 47, 76</td>
</tr>
<tr>
<td></td>
<td>Pareto efficient .......... 47, 76</td>
</tr>
<tr>
<td></td>
<td>weakly Pareto efficient ...... 47</td>
</tr>
<tr>
<td></td>
<td>Payoff functions ........... 154</td>
</tr>
<tr>
<td></td>
<td>Potential games ............ 173</td>
</tr>
<tr>
<td></td>
<td>Potential ................ 173</td>
</tr>
<tr>
<td></td>
<td>Preference relations ...... 45, 71</td>
</tr>
<tr>
<td></td>
<td>Archimedean property ...... 45, 71</td>
</tr>
<tr>
<td></td>
<td>Constant marginal utility .. 45, 80</td>
</tr>
<tr>
<td></td>
<td>Continuity in money .......... 45, 71</td>
</tr>
<tr>
<td></td>
<td>Monotonicity in money ........ 45, 71</td>
</tr>
<tr>
<td></td>
<td>Weak desirability for goods .. 45</td>
</tr>
<tr>
<td></td>
<td>Price equilibrium .......... 61, 79, 144</td>
</tr>
<tr>
<td></td>
<td>Budget constraint .......... 61, 79, 144</td>
</tr>
<tr>
<td></td>
<td>Maximality condition ....... 61, 79, 144</td>
</tr>
<tr>
<td></td>
<td>Price density ............. 144</td>
</tr>
<tr>
<td></td>
<td>Price vector ............... 61, 79</td>
</tr>
<tr>
<td></td>
<td>Prisoner’s dilemma ........ 4</td>
</tr>
<tr>
<td></td>
<td>Processing games .......... 137</td>
</tr>
<tr>
<td></td>
<td>with Shared Interest ....... 149</td>
</tr>
<tr>
<td></td>
<td>Processing problems ........ 133</td>
</tr>
<tr>
<td></td>
<td>Profit-making ............. 118</td>
</tr>
<tr>
<td></td>
<td>Prosperity property ........ 18</td>
</tr>
<tr>
<td></td>
<td>Closedness ................. 18</td>
</tr>
<tr>
<td></td>
<td>Monotonicity ............... 18</td>
</tr>
<tr>
<td></td>
<td>Weak prosperity property ... 18</td>
</tr>
<tr>
<td></td>
<td><strong>Q</strong></td>
</tr>
<tr>
<td></td>
<td>Quasi-linear utilities .... 45, 81, 143</td>
</tr>
<tr>
<td></td>
<td><strong>R</strong></td>
</tr>
<tr>
<td></td>
<td>Rationality ................ 1</td>
</tr>
<tr>
<td></td>
<td>Coalition rationality ...... 15</td>
</tr>
<tr>
<td></td>
<td>Individual rationality ..... 15, 48</td>
</tr>
<tr>
<td></td>
<td>Self-enforcingness .......... 157</td>
</tr>
<tr>
<td></td>
<td>Reachability ............... 161</td>
</tr>
<tr>
<td></td>
<td>Reactive bargaining set ... 20, 50</td>
</tr>
<tr>
<td></td>
<td>Reallocation .............. 46</td>
</tr>
<tr>
<td></td>
<td>$S$-reallocation ........... 46, 143</td>
</tr>
<tr>
<td></td>
<td>$S$-redistribution .......... 46, 143</td>
</tr>
<tr>
<td></td>
<td>(strong) Core reallocation .. 47, 143</td>
</tr>
<tr>
<td></td>
<td>Reasonableness ............ 117</td>
</tr>
<tr>
<td></td>
<td>On both sides ............. 117</td>
</tr>
<tr>
<td></td>
<td>On one side ............... 119</td>
</tr>
<tr>
<td></td>
<td>Reduced game property ...... 103</td>
</tr>
<tr>
<td></td>
<td>Regret-functions .......... 155</td>
</tr>
<tr>
<td>Index</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>---</td>
</tr>
<tr>
<td>Reservation values</td>
<td>46, 143</td>
</tr>
<tr>
<td>Rest-point</td>
<td>161</td>
</tr>
<tr>
<td><strong>S</strong></td>
<td></td>
</tr>
<tr>
<td>Semireactive bargaining set</td>
<td>19, 51</td>
</tr>
<tr>
<td>Smith's rule</td>
<td>136</td>
</tr>
<tr>
<td>Social welfare</td>
<td>47, 81</td>
</tr>
<tr>
<td>Social welfare condition</td>
<td>62</td>
</tr>
<tr>
<td>Stochastic redistribution</td>
<td>62</td>
</tr>
<tr>
<td>Solution concept</td>
<td>14, 155</td>
</tr>
<tr>
<td>Solution rule</td>
<td>117</td>
</tr>
<tr>
<td>Stand-alone costs</td>
<td>101</td>
</tr>
<tr>
<td>Standard solution</td>
<td>6, 122</td>
</tr>
<tr>
<td>Strategic equivalence</td>
<td>177</td>
</tr>
<tr>
<td>Strategies</td>
<td>154</td>
</tr>
<tr>
<td>Completely mixed</td>
<td>157</td>
</tr>
<tr>
<td>Mixed</td>
<td>154</td>
</tr>
<tr>
<td>Pure</td>
<td>154</td>
</tr>
<tr>
<td>Subsidy</td>
<td>140</td>
</tr>
<tr>
<td>Symmetric TU-games</td>
<td>25</td>
</tr>
<tr>
<td><strong>T</strong></td>
<td></td>
</tr>
<tr>
<td>Tax deposit</td>
<td>140</td>
</tr>
<tr>
<td>Total abundance condition</td>
<td>48</td>
</tr>
<tr>
<td>Trees</td>
<td>98</td>
</tr>
<tr>
<td>Branch</td>
<td>98</td>
</tr>
<tr>
<td>Children</td>
<td>98</td>
</tr>
<tr>
<td>Edges</td>
<td>98</td>
</tr>
<tr>
<td>Leaf</td>
<td>98</td>
</tr>
<tr>
<td>Maximal trunk</td>
<td>98</td>
</tr>
<tr>
<td>Nodes</td>
<td>98</td>
</tr>
<tr>
<td>Path</td>
<td>98</td>
</tr>
<tr>
<td>Predecessor</td>
<td>98</td>
</tr>
<tr>
<td>Root</td>
<td>98</td>
</tr>
<tr>
<td>Trunk</td>
<td>98</td>
</tr>
<tr>
<td>Trees with revenues</td>
<td>98</td>
</tr>
<tr>
<td>Inhabitants</td>
<td>98</td>
</tr>
<tr>
<td>Population distribution</td>
<td>98</td>
</tr>
<tr>
<td>Reduced tree with revenues</td>
<td>101</td>
</tr>
<tr>
<td>Revenues</td>
<td>98</td>
</tr>
<tr>
<td>TU-games</td>
<td>14</td>
</tr>
<tr>
<td>Balancedness</td>
<td>16</td>
</tr>
<tr>
<td>Convexity</td>
<td>17</td>
</tr>
<tr>
<td>Superadditivity</td>
<td>16</td>
</tr>
<tr>
<td>Total balancedness</td>
<td>16</td>
</tr>
<tr>
<td><strong>U</strong></td>
<td></td>
</tr>
<tr>
<td>Urgencies</td>
<td>136</td>
</tr>
<tr>
<td>Utility functions</td>
<td>73</td>
</tr>
<tr>
<td><strong>W</strong></td>
<td></td>
</tr>
<tr>
<td>Walrasian equilibrium</td>
<td>61</td>
</tr>
<tr>
<td>Weakly dominant (n)-partition</td>
<td>80</td>
</tr>
<tr>
<td><strong>Z</strong></td>
<td></td>
</tr>
<tr>
<td>Zero-sum games</td>
<td>168</td>
</tr>
<tr>
<td>Min-max value</td>
<td>168</td>
</tr>
</tbody>
</table>
Samenvatting

 Dit proefschrift behandelt een aantal onderwerpen binnen de speltheorie. Deze tak van wiskunde houdt zich bezig met het formeel modelleren van conflictsituaties en het eventueel aandragen van oplossingen. Een conflictsituatie kan een klassiek spel zijn, maar ook een economische situatie waarin vele individuen participeren, ieder met zijn eigen voorkeuren, capaciteiten, kennis en gedrag. Van oudsher wordt er een onderscheid gemaakt tussen non-coöperatieve en coöperatieve speltheorie. De laatst genoemde tak binnen de speltheorie bestudeert situaties waarin de betrokken partijen (de spelers) tot bindende afspraken kunnen komen, terwijl non-coöperatieve speltheorie conflictsituaties bestudeert met een meer competitief karakter.

Een veel gebruikte manier om een coöperatieve situatie te modelleren is aan de hand van een spel met overdraagbaar nut, een zogenaamd TU-spel. Dit eenvoudige model heeft als belangrijkste eigenschap dat de betrokken spelers de vruchten van hun samenwerking gelijkwaardig waarderen. Dit is alleszins een redelijke aanname wanneer de te verdelen opbrengst in termen van geld kan worden uitgedrukt en alle betrokkenen dezelfde waardering voor geld hebben.

Naast het modelleren is het geven van oplossingsconcepten een belangrijk aspect van de speltheorie. Bijvoorbeeld in de coöperatieve speltheorie is de vraag of en hoe de spelers samenwerken onlosmakelijk verbonden met de vraag wat die samenwerking uiteindelijk voor iedereen oplevert. Anders gezegd, het geven van een oplossingsregel die door alle spelers als ‘eerlijk’ wordt beschouwd kan als een motivatie dienen om samen te werken. In het begin van het tweede hoofdstuk van dit proefschrift worden enkele oplossingsconcepten voor TU-spellen herhaald. Een van deze concepten is de core. Kortweg is een voorgestelde verdeling van de totale opbrengst een core element als er geen groep van spelers bestaat die zich kan verbeteren. Hoewel de core een heel natuurlijk oplossingsconcept lijkt te zijn, is het vrij eenvoudig om voorbeelden van TU-spelen te maken waarvoor er geen core element bestaat.

Een ander bekend oplossingsconcept, waarvoor bovenstaand probleem niet opgaat, is de bargaining set. Deze verzameling van oplossingsregels is gebaseerd op het idee dat alvorens de spelers tot een verdeling van de opbrengst komen, er voorstellen worden gedaan waarop gemotiveerde bezwaren gemaakt kunnen worden. Alleen allocaties waarvoor al deze bezwaren verworpen worden, zijn elementen van de bargaining set. Twee andere oplossingsconcepten die in verband staan met bargaining set zijn de reactive bargaining set en de semireactive bargaining set. Beide oplossingsconcepten zijn ook gebaseerd op het idee dat ieder gemotiveerd bezwaar van een speler tegen een andere speler, door de laatste verworpen moeten kunnen worden. De verschillen liggen in de informatie die een speler heeft over de bezwaren van zijn opponent, alvorens hij zo’n bezwaar moet weerleggen. Omdat core elementen in ieder van deze drie bargaining sets liggen en de bargaining sets voor TU-spelen
Samenvatting

niet leeg zijn, kan ieder van deze verzamelingen van oplossingsregels als een uitbreiding van
de core gezien worden. Dit leidt tot de vraag: ‘Wanneer vallen de core en een van deze
bargaining sets samen?’.

Deze vraag wordt in Hoofdstuk 2 bestudeert voor *symmetrische* TU-spelen. Hoewel voor
dezelfde eenvoudige klasse van TU-spelen een bargaining set element niet de meest voor de hand
liggende oplossingsregel is, geeft deze klasse van spelen een goed beeld van de verschillen
tussen deze drie bargaining sets. Daarnaast wordt een symmetrisch TU-spel gebruikt om een
verrasende en opmerkelijke relatie tussen de bargaining set en de core te laten zien. Een
dertien-persoons symmetrisch TU-spel laat zien dat de core en bargaining set samenvallen
voor alle mogelijke waarden van de totaal te verdelen opbrengst behalve voor *precies één
waarde*. Daar is de bargaining set gelijk aan de core verenigd met een eindig aantal allocaties
buiten de core.

In Hoofdstuk 3 wordt coöperatieve speltheorie toegepast om *ruil-economieën met ondeelbare
goederen en geld* te bestuderen. In deze economieën hebben spelers, ook wel *agenten* ge-
noemd, een aantal ondeelbare goederen plus een hoeveelheid geld in hun bezit, die ze met
echter kunnen ruilen. In het grootste gedeelte van dit hoofdstuk worden de drie bargaining
sets uit Hoofdstuk 2 gebruikt om tot ‘goede’ verdelingen van de goederen en het geld te
komen. In deze context zijn ieder van de drie bargaining sets *niet* leeg zodra iedere agent
genoeg geld heeft, dat wil zeggen de ruil-economie voldoet aan de zogenaamde *Total Abun-
dance (TA)* conditie, én de preferenties van de agenten gerepresenteerd kunnen worden met
*quasi-lineaire* nutsfuncties. Voorbeelden tonen aan dat in dit resultaat géén van deze voor-
waarden zonder meer kunnen worden weggelaten.

Dit hoofdstuk stelt zich ook ten doel om een één-op-één correspondentie te maken tussen
oplossingsconcepten voor TU-spelen en oplossingsconcepten voor ruil-economieën. Hoewel,
on het eerste gezicht de ondeelbaarheid van de goederen moeilijk lijkt te rijmen met het
overdraagbaar nut van TU-spelen, kan er wel degelijk een link worden gelegd tussen beide
modellen. Zodra in een ruil-economie de nutsfuncties quasi-linear zijn en er wordt boven-
dien aan de TA-conditie voldaan, dan kan ieder (semi)reactive bargaining set element in de
economie worden vertaald in een (semi)reactive bargaining set element van het bijbehorende
TU-spel, en omgekeerd. Een soort gelijke bewering geldt voor de core zodra het bijbehorende
spel *gebalanceerd* is. Voor de bargaining set gaat deze redenering niet op. De reden hiervoor
ligt in het feit dat een bargaining set element in de ruil-economie niet *individueel rationeel* is.
Kortweg betekent dit dat in zo’n allocatie een agent een bundel krijgt die hij minder waardeert
dan zijn oorspronkelijke bundel. Allocaties in de overige twee bargaining sets zijn echter wel
*individueel rationeel*.

In de laatste paragraaf van dit hoofdstuk wordt een relatie gelegd tussen TU-spelen en het
bestaan van *prijs-evenwichten* in een ruil-economie met ondeelbare goederen en geld. Hier
wordt bewezen dat elk *niet-negatief* en *superadditief* TU-spel een ruil-economie met een
prijs-evenwicht genereert, zodra het spel een *niet-lege core* heeft.

Hoofdstuk 4 houdt zich ook bezig met *economieën met ondeelbare goederen en geld*. Echter,
in tegenstelling tot Hoofdstuk 3 hebben agenten nu geen initieel bezit. De vraag is wederom
of er ‘eerlijke/goede’ allocaties van de objecten en het geld bestaan. Omdat de nutsfuncties
hier *niet* meer quasi-linear zijn, wordt er niet gekeken naar oplossingsconcepten uit de the-
orie van TU-spelen. Het eerste normatieve concept voor een eerlijke verdeling waar naar gekeken wordt, is een allocatie die géén afgunst opwekt. Een allocatie wekt geen afgunst op, wanneer iedere agent zijn toegekende bundel het meest waardeert over alle andere bundels in de allocatie. Het belangrijkste resultaat van dit hoofdstuk zegt dat zo’n verdeling bestaat, zodra de preferentie relaties van de agenten continue en strikt monotoon in geld zijn en aan de archimedische eigenschap voldoen. Deze laatste eis zegt zoveel dat alle goederen door geld gecompenseerd kunnen worden. Voorbeelden laten zien dat de continuïteit alsmede de archimedische eigenschap van de preferentie relaties niet zomaar kunnen worden weggelaten voor de existentie van allocatie waarin niemand elkaar benijdt.

Een eenvoudig voorbeeld van een economie met twee agenten en twee ondeelbare goederen laat zien dat alle Pareto optimale allocaties afgunst kunnen opwekken. Een allocatie is Pareto optimaal wanneer géén agent zijn positie kan verbeteren zonder de positie van de overige agenten te schaden. Het niet compatibel zijn van deze twee begrippen laat dus zien dat om afgunst te vermijden een verdeling gekozen moet worden die niet optimaal is, in de zin dat iedere agent er op vooruit kan gaan door de goederen en het geld anders te verdelen.

**Hoofdstuk 5** bekijkt de situatie waarin bewoners van dorpen, die via een vast boom-netwerk met elkaar verbonden zijn, inkomsten krijgen zodra ze verbonden zijn met de wortel van deze boom. Hierbij kan bijvoorbeeld gedacht worden aan een irrigatie-systeem; de wortel is een meer en de takken van de boom zijn de kanalen in het systeem. Zodra een speler (bewoner van een dorp) over water beschikt (d.w.z. een (in)directe verbinding heeft met de wortel), kan hij zijn brood verdienen. De kanalen in het irrigatie-systeem brengen bepaalde kosten met zich mee (bijvoorbeeld reinigings-kosten). De vraag die er nu gesteld wordt is wat iedere speler van zijn inkomsten moet afgeven om samen de totale kosten van het netwerk te kunnen betalen.

Om dit probleem op te lossen wordt bij deze boom met inkomsten een coöperatief TU-spel geconstrueerd en wordt de nucleolus van dit spel als een oplossingsregel voorgesteld. Het grootste gedeelte van het hoofdstuk houdt zich bezig om aan de hand van een gegeven boom-netwerk en de individuele inkomsten van de spelers, de nucleolus op een efficiënte manier uit te rekenen. Daarnaast wordt deze oplossingsregel gekarakteriseerd. Dat wil zeggen er wordt een lijst van zeven ‘redelijke’ eigenschappen van oplossingsregels geïntroduceerd en bewezen dat de nucleolus de enige oplossingsregel is die aan alle eigenschappen voldoet. Ten slotte wordt bewezen dat de nucleolus van een boom met opbrengsten op drie verschillende manieren monotoon is. Ten eerste is de nucleolus kosten-monotoon, wat zoveel wil zeggen dat alle spelers meer betalen zodra de kosten van een tak toenemen. Aan de andere kant, wanneer de inkomsten van een speler toenemen, dan hoeft volgens de nucleolus iedere speler minder te gaan betalen. Met andere woorden, de nucleolus van een boom met opbrengsten is inkomsten-monotoon. De nucleolus is ook populatie-monotoon. Dit wil in feite zeggen dat iedere speler meer gaat betalen zodra een bewoner van een dorp het netwerk verlaat.

In Hoofdstuk 6 worden processing spelen geïntroduceerd. Deze TU-spelen worden gebruikt om de gezamenlijke kosten van een processing situatie eerlijk over de betrokkenen te verdelen.

In een processing situatie heeft iedere speler een individuele capaciteit om taken te verwerken. Bij een taak kan bijvoorbeeld gedacht worden aan het oplossen van een complex
Samenvatting

stelsel vergelijkingen via de computer en de capaciteit als reken snelheid van deze computer. Iedere speler is verantwoordelijk voor één taak. Gedurende iedere tijdseenheid dat een taak van een speler niet af is worden er kosten in rekening gebracht. Door eventueel samen te werken kunnen taken eerder afgerond worden en op deze manier kunnen dus kosten worden bespaard. Wanneer een groep spelers besluit om samen te werken, dan heeft deze een capaciteit ter grootte van alle individuele capaciteiten ter beschikking, om de taken van alle leden van de groep te voltooien. Hoewel er geen restricties zijn op welke manier de spelers dit zouden moeten doen, kunnen de minimale kosten gerealiseerd worden door op ieder tijdstip de volledige capaciteit voor precies één taak te gebruiken en de taken in een vaste volgorde af te handelen. Deze optimale volgorde kan worden bepaald via zogenaamde urgenties van de taken.

Gegeven hoe een groep spelers optimaal kan samenwerken in een processing situatie, is de volgende vraag hoe deze spelers de totale minimale kosten moeten verdelen. In dit hoofdstuk wordt een antwoord op deze vraag gegeven door voor iedere processing situatie een expliciete kosten-verdeling te definiëren die aan de core-condities van het bijbehorende processing spel voldoet. Deze allocatie van de totale kosten wordt verkregen door middel van een prijsevenwicht in een geschikt gekozen ruil-economie met land.

In Hoofdstuk 7 krijgen we te maken met non-coöperatieve speltheorie. Zoals eerder gezegd is het in deze tak van speltheorie niet mogelijk om bindende afspraken te maken en staan de spelers als ‘concurrenten’ tegenover elkaar. Een bekend model in non-coöperatieve speltheorie is het zogenaamde spel in normale vorm. In dit model participeren een eindig aantal spelers. Iedere speler beschikt over een eindig aantal acties. Als iedere speler, tegelijk en onafhankelijk van elkaar, één van zijn acties kiest, leidt dit tot een uitkomst en is het spel afgelopen. Iedere speler heeft zijn eigen waardering voor elk van de mogelijke uitkomsten.

Een belangrijk probleem in de non-coöperatieve speltheorie is welke actie een speler moet kiezen om voor hem de ‘beste’ uitkomst te realiseren. Omdat deze uitkomst niet alleen van zijn eigen acties af hangt, maar ook van de acties van zijn tegenstanders, is deze vraag verre van triviaal. Het misschien wel meest beroemde oplossingsconcept in non-coöperatieve speltheorie is de verzameling van Nash-evenwichten. Een Nash-evenwicht geeft aan iedere speler een actie met de eigenschap dat dit de beste actie is, gegeven de acties van de overige spelers. Anders gezegd, éénzijdig afwijken levert aan een speler geen hogere uitbetaling.

Een belangrijke vraag die ten grondslag ligt aan dit evenwicht is hoe spelers er toe komen om gezamenlijk zo’n Nash-evenwicht te spelen. Om deze vraag te beantwoorden is er in de literatuur een link gelegd tussen speltheorie en evolutieetheorie. In dit hoofdstuk wordt dit idee geformaliseerd aan de hand van een dynamisch selectie proces, gebaseerd op een stelsel differentiaal-vergelijkingen. De belangrijkste resultaten zeggen dat voor twee-persoons nul-som spelen en voor n-persoons potentiaal spelen dit proces naar de verzameling van Nash-evenwichten convergeert. Intuïtief zegt dit dat voor deze twee klasse van normale-vorm spelen een evolutioneer selectie proces bestaat dat naar Nash-evenwichten convergeert.
Marc Meertens was born on November 16th 1977 in Kerkrade, The Netherlands. In 1990, after elementary school, he went to Sint Antonius Doctor College in Kerkrade where he obtained his VWO diploma in 1996. In the same year he moved to Nijmegen to study Mathematics at the University of Nijmegen. During these studies he worked as a teaching assistant at the same university. In the beginning of 2000 he wrote his Master thesis entitled ‘Envy-free Allocations’ under supervision of Jos Potters and Hans Reijnierse. Shortly after finishing his Master thesis he took an internship at the statistical research department of UWV Cadans in Zeist, The Netherlands. At the end of 2001 he received his Master’s degree in Mathematics (cum laude).

While studying Mathematics his main interests included subjects in Game Theory. In 2001, he therefore assumed a Ph.D. position in Game Theory at the University of Nijmegen. Again the supervision was carried out by Jos Potters and Hans Reijnierse. During a period of four years, he worked in two research groups, in Nijmegen and Tilburg, and wrote articles with several co-authors on topics in Cooperative and in Non-cooperative Game Theory. Most of the results of this research can be found in this thesis.